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Zero-dimensional sheaves, group actions and blowups

Candidate:
Michele Graffeo

Advisor:
Ugo Bruzzo
Co-advisors:
Andrea T. Ricolfi

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MA PER SEGUIR VIRTUTE E CANOSCENZA
SISSA

To Irene and Moira

Abstract

The main objects of study of this thesis are 0-dimensional subschemes of affine spaces. More precisely, I have studied the following two aspects concerning them:

- the interaction between 0-dimensional subschemes and linear group actions on $\mathbb{A}^n$,
- the computation of the Behrend number of 0-dimensional schemes in order to better understand the Hilbert scheme of points.

In the first chapter of the thesis I have constructed the moduli spaces of certain G-equivariant coherent $\mathscr{O}_{\mathbb{A}^2}$-modules ($G$-constellations), introduced by Alastair Craw in 2001, which are stable with respect to a GIT stability condition. In addition, I studied the associated chamber decomposition giving an explicit combinatorial description of the chambers.

In the second part of the thesis I have computed, mostly applying techniques from toric geometry, the Behrend number of a large number of fat points of the affine plane. This invariant had been abstractly defined by Behrend in 2009, but even for a scheme with only one point the (few) existing methods to calculate it could not be applied.

The thesis is mainly based on the content of the following two preprints:

- *"Moduli spaces of $\mathbb{Z}/k\mathbb{Z}$-constellations over $\mathbb{A}^2$".* [30, 2022]
- *"On the Behrend function and the blowup of some fat points"*, with A. T. Ricolfi.[31, 2022]

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Introduction

The objects of interest in algebraic geometry are *algebraic varieties*, i.e. common zero loci of collections of polynomials. The easiest algebraic variety is just a d-tuple of distinct points, say in some fixed affine space $\mathbb{A}^n$. More generally, one is interested in 0-*dimensional subschemes* $Z \subset \mathbb{A}^n$, obtained as spectra of semilocal artinian $\mathbb{C}$-algebras of finite type (we say *fat point* in the local case). Such schemes are called *smoothable* when they are a limit of (a disjoint union of) distinct points. Here, by limit, we mean that there exist a flat family of 0-dimensional subschemes of $\mathbb{A}^n$, $\mathcal{X} \xrightarrow{\varphi} B$, with special fibre the scheme Z and with general fibre an d-tuple of distinct points.

However, not all 0-dimensional schemes are smoothable. In technical terms, this is captured by the reducibility of the moduli space parametrising 0-dimensional subschemes $Z \subset \mathbb{A}^n$ of fixed length d. This space, introduced by Grothendieck, is known as the *Hilbert scheme of d points* $\mathrm{Hilb}^d(\mathbb{A}^n)$. Even in low dimensions, this moduli space is very singular and complicated. For instance, although it is known (see for instance [11]) that, for $n \geq 4$, $\mathrm{Hilb}^d(\mathbb{A}^n)$ is irreducible for $d \leq 7$ and reducible for $d \geq 8$, the irreducibility of $\mathrm{Hilb}^d(\mathbb{A}^3)$ for $11 < d < 78$ (see [18] and the references therein) is currently unknown. Likewise, it is not known, for any $d \geq 78$, whether $\mathrm{Hilb}^d(\mathbb{A}^3)$ has non-reduced components. Finally, the existence of a new component, for $d \geq 78$, has no constructive proof, so there are no explicit examples of new components as well as of non-smoothable fat points of embedding dimension 3.

Unluckily, as observed for example in [41], being reducible is only one of the possible pathologies that occur on the Hilbert scheme of points. Indeed, any kind of singularity occurs.

Another area of algebraic geometry in which the study of 0-dimensional schemes plays a central role is Representation Theory, in particular, the theory of finite subgroups of $\mathrm{GL}(n,\mathbb{C})$. When a finite linear group G acts on $\mathbb{A}^n$, there exists a G-equivariant notion of 0-dimensional subscheme of $\mathbb{A}^n$, namely that of G-cluster (see Definition 1.0.1). In this context, it is possible to also introduce a notion of G-Hilbert scheme G-$\mathrm{Hilb}(\mathbb{A}^n)$ (see Definition 1.0.4), i.e. the fine moduli space of G-clusters. When $G < \mathrm{SL}(3,\mathbb{C})$ the space G-$\mathrm{Hilb}(\mathbb{A}^3)$ has several nice properties (see [9]) and it has been studied in many areas of Mathematics and Mathematical Physics.

Although, thanks to the results in [9], today a lot is known about G-$\mathrm{Hilb}(\mathbb{A}^3)$ when $G < \mathrm{SL}(3,\mathbb{C})$, in some instances this space has not been concretely constructed and its geometry has not been fully studied. Equally fascinating is the case $n \geq 4$, where much less is known, and also many definitions should be revised. For instance, when $n \geq 4$, it is not in general clear how the McKay correspondence should be formulated.

Thanks to a GIT argument (see [9, 15, 30]), G-Hilb can be interpreted as the moduli space of certain stable G-equivariant coherent sheaves, namely the G-constellations (see Definition 1.0.8), with respect to a GIT stability condition. This allows one, as the stability condition varies, to build many other moduli spaces that share several nice properties with G-Hilb. Yet much less is known about them.

Moduli spaces of G-constellations

One of the widest research areas in algebraic geometry is the resolution of singularities. In particular, when $X = \mathbb{A}^n/G$ for some finite subgroup $G < \mathrm{SL}(n,\mathbb{C})$, one can ask if a crepant resolution (see Definition 0.3.1 and Section 0.5) $\varphi : Y \to X$ exists and, if it does, what is its relation with the group G. The first answer to those questions was given, in dimension 2, by the so-called McKay correspondence for A-D-E singularities (see [47, 29]). Nowadays, crepant resolutions of singularities of the form $\mathbb{A}^n/G$, where $G \subset \mathrm{SL}(n,\mathbb{C})$ is a finite subgroup, appear in several fields of Algebraic Geometry and Mathematical Physics, for example see [10, 40, 61] and the references therein.

In general, crepant resolutions may not exist. Nevertheless, it is known that they exist in dimension 2 and 3: see [19] for dimension 2, and see Roan [63, 64], Ito [38] and Markushevich [50] for dimension 3. In particular, the 3-dimensional case was solved by a case by case analysis, taking advantage of the fact that the conjugacy classes of finite subgroups of $\mathrm{SL}(3,\mathbb{C})$ had been listed, for example in [72].

More recently, in [9], Bridgeland, King and Reid proved in one shot that a resolution always exists in dimension 3. The resolution that they proposed is made in terms of G-clusters, i.e. G-equivariant 0-dimensional subschemes Z of $\mathbb{A}^n$ such that $H^0(Z,\mathscr{O}_Z) \cong \mathbb{C}[G]$ as G-modules (Definition 1.0.1). In particular, in [9] it was proved that there exists a crepant resolution

$$G\text{-}\mathrm{Hilb}(\mathbb{A}^3) \to \mathbb{A}^3/G$$

where G-Hilb$(\mathbb{A}^3)$ is the irreducible component of the fine moduli space of G-clusters containing free orbits. Notice that this result had already been obtained for abelian actions by Nakamura in [52].

In particular, G-Hilb$(\mathbb{A}^n)$ is a closed G-invariant subscheme of the Hilbert scheme of $|G|$ points in $\mathbb{A}^n$. The existence, for $n = 2,3$, of a crepant resolution of singularities $\varphi : G\text{-}\mathrm{Hilb}(\mathbb{A}^n) \to X$ was proven in [9] where the authors also showed that there is an equivalence of categories between the derived category of G-Hilb$(\mathbb{A}^n)$ and the derived category of coherent G-sheaves on $\mathbb{A}^n$. Nonetheless, it is well known that, in higher dimensions, a crepant resolution may not exist and, even if it exists, it may not be given by G-Hilb$(\mathbb{A}^n)$.

In [52], Nakamura also introduced, in a similar way, the notion of $G\text{-}\mathrm{Quot}^{\mathscr{F}}(\mathbb{A}^3)$ and he asked for which coherent G-sheaves $\mathscr{F} \in \mathrm{Ob}\,\mathrm{Coh}(\mathbb{A}^3)$ the variety $G\text{-}\mathrm{Quot}^{\mathscr{F}}(\mathbb{A}^3)$ is a projective crepant resolution of $\mathbb{A}^3/G$.

Alastair Craw in his PhD thesis [14] generalized the notion of G-cluster to the notion of G-constellation, i.e. 0-dimensional G-equivariant coherent $\mathscr{O}_{\mathbb{A}^n}$-module $\mathscr{F}$ such that $H^0(\mathbb{A}^n,\mathscr{F}) \cong \mathbb{C}[G]$ as representations. Moreover, he introduced a GIT stability notion, in the sense of King, on the category of G-constellations. Afterwards, in [15], Craw and Ishii observed that, if $G < \mathrm{SL}(3,\mathbb{C})$ is abelian, then, for a generic stability condition θ, there exists a crepant resolution of singularities

$$\mathscr{M}_\theta \to \mathbb{A}^3/G$$

where $\mathscr{M}_\theta$ is the irreducible component of the fine moduli space of θ-stable G-constellations that contains free orbits. Moreover, in the same paper, they proved that all crepant resolutions are of the above form and they differ from each other by a series of flops induced by wall-crossing in the space of generic stability conditions and they conjectured that the same is true for any finite subgroup of $\mathrm{SL}(3,\mathbb{C})$.

Very recently, a preprint [71] containig the proof of Craw-Ishii's conjecture has appeared. Unfortunately, there was no time to connect it with this thesis. We will briefly comment on this proof in Section 1.7.

It was observed in [15] that the results in [9] imply that the space of generic stability conditions Θ^{gen} is a disjoint union of connected components called chambers. Moreover, in each chamber C, the notion of stability is constant, i.e. for any $\theta,\theta' \in C$, a G-constellation is θ-stable if and only if it is θ'-stable. Therefore, there is a canonical isomorphism $\mathscr{M}_\theta$ and $\mathscr{M}_{\theta'}$ for all $\theta,\theta' \in C$ and one can write $\mathscr{M}_C$ instead of $\mathscr{M}_\theta$.

In the first part of the thesis, we will focus on the 2-dimensional abelian case, i.e. the case when $G < \mathrm{SL}(2,\mathbb{C})$ is a finite abelian, and hence cyclic, subgroup. In the literature the singularity $\mathbb{A}^2/G$ is called an $A_{|G|-1}$ singularity. This case is particularly simple from the point of view of the resolution because we know, from classical surface theory, that there is a unique minimal crepant resolution. Therefore, all the moduli spaces $\mathscr{M}_\theta$ are isomorphic as quasi-projective varieties. As a consequence, in order to distinguish two chambers it is enough to study their universal families $\mathscr{U}_C \in \mathrm{Ob}\,\mathrm{Coh}(\mathscr{M}_C \times \mathbb{A}^2)$.

In Chapter 1, which mostly follows [30], we give an explicit combinatorial description of the moduli spaces $\mathscr{M}_\theta$, in the 2-dimensional abelian case, in terms of the chambers decomposition studied by Craw and Ishii. In particular, this description answers, in dimension 2, the question raised by Nakamura in [52] about G-$\mathrm{Quot}^{\mathscr{F}}$. The key tool used is the following generalisation of [29, Proposition 2.4].

Theorem A ([30], Theorem 1.5.2)**.** *Let $\pi : \mathbb{A}^2 \to X$ be the projection map where $X = \mathbb{A}^2/G$ and $G < \mathrm{SL}(2,\mathbb{C})$ is an abelian finite subgroup and let $\varepsilon : Y \to X$ be the crepant resolution of singularities. If $\mathscr{K} \subset \mathscr{O}_{\mathbb{A}^2}$ is a coherent (G-invariant) monomial ideal sheaf, then the $\mathscr{O}_Y$-module*

$$\varepsilon^*\pi_*\mathscr{K} \,/\, \mathrm{Tor}_{\mathscr{O}_Y}\, \varepsilon^*\pi_*\mathscr{K}$$

is locally free of rank $|G|$.

The usefulness of Theorem A consists for example in the fact that, analogously to [29, Proposition 2.4], it provides the McKay correspondence and a generalisation of it would help to get higher dimensional versions of the McKay correspondence. The proof of Theorem A will be constructive, meaning that we will give a commutative algebra construction that allows one to write an explicit formula for the tautological bundle

$$\mathscr{R}_\theta \in \mathrm{Ob}\,\mathrm{Coh}(\mathscr{M}_\theta),$$

i.e. the pushforward of the universal family $\mathscr{U}_\theta \in \operatorname{Ob}\operatorname{Coh}(\mathscr{M}_\theta \times \mathbb{A}^2)$ via the first projection. This construction can be easily implemented using some software such as Macaulay2 [32]. As a consequence of Theorem A, we have obtained the following theorem, which, having a constructive proof, allows one to build explicitly the objects mentioned in the statement.

Theorem B ([30], Corollary 1.6.8)**.** *Given G as in Theorem A and a generic stability condition θ, there exists one (in fact infinitely many) G-invariant coherent ideal sheaf $\mathscr{K} \subset \mathscr{O}_{\mathbb{A}^2}$ such that $\mathscr{M}_\theta$ can be identified with a closed G-invariant subvariety of $\operatorname{Quot}^{\mathscr{K}}_{|G|}(\mathbb{A}^2)$, where,*

$$\operatorname{Quot}^{\mathscr{K}}_{|G|}(\mathbb{A}^2) = \left\{ \mathscr{K} \twoheadrightarrow \mathscr{F} \;\middle|\; \mathscr{F} \in \operatorname{Ob}\operatorname{Coh}(\mathbb{A}^2),\ \dim H^0(\mathbb{A}^2, \mathscr{F}) = |G| \right\} / \sim .$$

The theory developed to get the above result also allowed us to detect a special collection S of chambers, the set of *simple chambers*. They have the property that any irreducible G-constellation belongs to, at least, one simple chamber. The set S is defined in terms of the toric G-constellations in $\mathscr{M}_\theta$ for θ generic and its cardinality is computed in Section 1.4. The following statement collects the mentioned results.

Theorem C ([30], Remark 1.4.16, Theorems 1.3.17 and 1.4.15)**.** *If $G < \mathrm{SL}(2,\mathbb{C})$ is an abelian subgroup of cardinality k, there are $k \cdot 2^{k-1}$ isomorphism classes of irreducible toric G-constellations and*

- *the space of generic stability conditions is the disjoint union of $k!$ chambers,*
- *the set of simple chambers has cardinality $k \cdot 2^{k-2}$.*

The first point in Theorem C can be also recovered, via different arguments, from the theory developed by Kronheimer in [47] (See also [12, Chapter 3-§3] for the algebraic interpretation), but the approach to the abelian case here is different and it helps to prove the other results.

In order to prove Theorem C, we will give an exhaustive combinatorial description of the toric points of the spaces $\mathscr{M}_\theta$ in terms of combinatorial objects called skew Ferrers diagrams. Such diagrams are standard tools in many branches of mathematics, e.g. enumerative geometry, group theory, commutative algebra etc (for example [8, 27, 51]). In order to define simple chambers, we will need to construct chamber stairs (Definition 1.4.2), combinatorial objects that we will use to encode all the data of a chamber C.

It is remarkable that, even if Theorem A is proven under the 2-dimensional and abelian hypotheses, in all the computational examples the statement happens to be true even in more general cases. For instance, we tested via Macaulay2 [32] many 2-dimensional bidihedral cases. In dimension 3 the situation is more delicate since there are several crepant resolutions. Nevertheless, choosing the coherent ideal G-sheaf appropriately gives similar results, again for specific examples (see Remark 1.6.3). In general, the case when $G < \mathrm{SL}(3,\mathbb{C})$ is non-abelian is still largely unknown. Moreover, even in the abelian case, it is not clear for which coherent G-sheaves $\mathscr{F}$ the variety G-$\operatorname{Quot}^{\mathscr{F}}(\mathbb{A}^2)$ should be isomorphic to $\mathscr{M}_\theta$ for some generic θ.

In order to investigate this situation and to generalize Theorem A to a more general setting, we began to study the action of the Klein group $H_{168} \cong \mathbb{P}\mathrm{SL}(2,\mathbb{F}_7)$ on $\mathbb{A}^3$. In particular, we built a crepant resolution (see Section 1.8) alternative to the one constructed by Markushevich in [50]. Comparing them will help in the construction of all the other resolutions and to construct explicitly the moduli spaces of H_{168}-constellations.

Another possible way to explicitly construct moduli spaces of G-constellations, for G non-abelian should be to combine the ideas in [53, 54] with the approach of Section 1.5.

The Behrend number of fat points via blowups

The Hilbert scheme of d points in the affine space $\mathbb{A}^n$ has been defined in [34] and it is a central object in modern algebraic geometry. However, not so much is known about it.

In dimension 2, Fogarty proved, in [23], that $\mathrm{Hilb}^d(\mathbb{A}^2)$ is a smooth quasi-projective variety and that the Hilbert-Chow morphism

$$\mathrm{Hilb}^d(\mathbb{A}^2) \to \mathrm{Sym}^d(\mathbb{A}^2)$$

is a resolution of singularities. Another important result is the connectedness theorem by Hartshorne in [35], which states that any Hilbert scheme (over a connected variety) is connected. In dimension $n > 2$ the situation is more complicated. For instance, it was observed that $\mathrm{Hilb}^d(\mathbb{A}^n)$ is singular for any $d \geq 4$.

One can also consider the *punctual Hilbert scheme*, i.e. the closed quasi-projective subvariety $\mathrm{Hilb}^d(\mathbb{A}^n)_0 \subset \mathrm{Hilb}^d(\mathbb{A}^n)$ that parametrises fat points supported at the origin $0 \in \mathbb{A}^n$. Clearly, one can acquire information about $\mathrm{Hilb}^d(\mathbb{A}^n)$ from $\mathrm{Hilb}^d(\mathbb{A}^n)_0$. For instance, they may share irreducible components (see [11]). Iarrobino (in [36]) and Briançon (in [8]) independently proved that the punctual Hilbert scheme can be stratified according to the Hilbert–Samuel function of fat points and they explicitly described this stratification in some instances. Moreover, in [37], Iarrobino proved, via a dimension argument, that $\mathrm{Hilb}^{78}(\mathbb{A}^3)$ is reducible. His proof passes trough the reducibility of $\mathrm{Hilb}^{78}(\mathbb{A}^3)_0$. He also pointed out that there is no reason why 78 should be the smallest number with this property. Unfortunately, the stratification provided by the Hilbert–Samuel function did not help us to get new answers about the many questions concerning Hilbert schemes of points. Therefore, a good idea might be to focus on some other invariant. One possibility is given by the Behrend function.

In general, given a scheme X of finite type over $\mathbb{C}$, the *Behrend function* (Section 2.1) is a constructible function $\nu_X : X(\mathbb{C}) \to \mathbb{Z}$ intrinsically attached to the scheme X. When X is a fat point, i.e. when X consists, topologically, of just one point, the Behrend function is, in fact, a number. Moreover, (Section 2.1) my coauthor and supervisor Andrea T. Ricolfi (SISSA) and I observed that, in this case, the Behrend number can be computed as the sum of the multiplicities of some effective divisors. Thus, $\nu_Z > 0$ for all fat point Z.

Currently, basically no method is known for calculating the Behrend function of fat points. Therefore, it is possible to calculate this invariant only in very simple cases such as the local complete intersection case. Since we expect this function to help us to better understand the behavior of the punctual Hilbert scheme, we have decided to evaluate it (see [31]) for the fat points of the affine spaces, in particular $\mathbb{A}^2$ and $\mathbb{A}^3$.

In the paper [31] we have developed an effective technique to compute the Behrend function of a large class of *fat points*.

As a side result, we were able to describe the geometry of many surfaces constructed as blowup of the affine plane at reducible and non-reduced centres. The same kind of analysis, in dimension 3, is much more complicated. In theory, it is fully described by the MMP theory (see [46, Chapter 2] for an introduction), but, in practice, many areas are still to be investigated. Some examples of similar birational constructions, in dimension 3, will be given in Section 1.8 and Appendix A.

We mostly focus on the case where X sits inside the affine plane $\mathbb{A}^2$, i.e. it has embedding dimension 1 or 2. Such planar situation, arguably the easiest one, already presents (interesting) technical difficulties, confirming that the Behrend function is a subtle invariant of a scheme, no matter how many points the scheme has! For instance, it is easy to observe that

$$\nu_X = \mathrm{length}(X)$$

in many *special* cases (see Section 2.2.2 for several examples), but in general the length of the fat point, namely the number $\mathrm{length}(X) = \chi(\mathscr{O}_X) = h^0(\mathscr{O}_X)$, is not equal to ν_X, and is neither a lower bound nor an upper bound for ν_X. As an example (cf. Example 2.2.7 for more details), consider the ideal $\mathfrak{m} = (x, y) \subset \mathbb{C}[x, y]$. Then, if $X_d = \operatorname{Spec}\mathbb{C}[x, y]/\mathfrak{m}^d$ for $d > 1$, one has

$$\nu_{X_d} = d < \frac{d(d+1)}{2} = \mathrm{length}(X_d).$$

Our main technique is a fine analysis of the multiplicities of the components of the exceptional divisor of the blowup $\mathrm{Bl}_X \mathbb{A}^N$, where $X \hookrightarrow \mathbb{A}^N$ is a given fat point. These multiplicities add up to ν_X by Lemma 2.2.2.

The following result is obtained combining toric geometry techniques with a deep analysis of the blowups of $\mathbb{A}^2$ along the given ideals.

Theorem D (Theorems 2.3.11 and 2.3.13)**.** *Let $1 \le i_1 < \cdots < i_s$ be a strictly increasing sequence of positive integers. Consider the ideal $K = \prod_{1\le k\le s}(x + f(y)) + \mathfrak{m}^{i_k} \subset \mathbb{C}[x, y]$, where $f(y) \in \mathbb{C}[y]$ has degree smaller than i_s. Then K has colength $\sum_{1\le k\le s}\sum_{1\le j\le k} i_j$, and its Behrend number is*

$$\nu_{\mathbb{C}[x,y]/K} = \sum_{k=1}^{s}\sum_{j=1}^{k} i_j + \sum_{j=1}^{s-1} i_j(s-j).$$

In particular, when $i_k = k$ for all $k = 1, \ldots, s$, the ideal $K_s = \prod_{1\le k\le s}(x + f(y)) + \mathfrak{m}^k$ has colength $\binom{s+2}{3}$, and its Behrend number is

$$\nu_{\mathbb{C}[x,y]/K_s} = \frac{s(s+1)(2s+1)}{6}.$$

Ideals as in the statement of Theorem D are called *towers* in the thesis (cf. Definition 2.3.4), and they are called *complete towers* when $i_k = k$ for all k. In Theorem 2.3.17 we give a formula for the Behrend number of a product of two complete towers; in Section 2.4 we examine the case of arbitrary finite products of towers, and we present an algorithm to compute the Behrend number also in this case.

Theorem D covers a large class of ideals, including *some* monomial ideals. We now present a few more explicit formulas.

If a monomial ideal $I \subset \mathbb{C}[x, y]$ is *normal* (which means that $\mathrm{Bl}_I \mathbb{A}^2$ is normal, cf. Section 0.10), then, by Corollary 2.6.11, it factors uniquely as a product

$$I = \prod_{k=1}^{t} \mathfrak{n}_{\alpha_k,\beta_k}^{\delta_k}, \qquad (\triangle)$$

where $\mathfrak{n}_{\alpha,\beta} \subset \mathbb{C}[x,y]$ denotes the normalisation of the ideal (x^α, y^β) and $\gcd(\alpha_k,\beta_k)=1$ for all $k=1,\ldots,t$.

The decomposition ($\triangle$) can be seen as an *explicit* instance of a more general 'unique factorisation theory' originally developed by Zariski [73] and more recently by Lipman (see [48, Section V] and the references therein).

Thanks to the following explicit result, we know the Behrend number of $\mathfrak{n}_{\alpha,\beta}$.

Theorem E (Theorem 2.6.5)**.** *Let $\alpha,\beta>0$ be two positive integers, and let $\mathfrak{n}_{\alpha,\beta} \subset \mathbb{C}[x,y]$ be the normalisation of the ideal (x^α, y^β). Then,*

$$\nu_{\mathbb{C}[x,y]/\mathfrak{n}_{\alpha,\beta}} = \frac{\alpha\cdot\beta}{\gcd(\alpha,\beta)}.$$

One can describe thoroughly the blowup $\mathrm{Bl}_I \mathbb{A}^2$ along a normal monomial ideal as in Equation ($\triangle$) via toric geometry (cf. Corollary 2.6.10), and this allows one to generalise the identity in Theorem E to cover all normal monomial ideals in $\mathbb{C}[x,y]$.

In fact, the existence of the factorisation ($\triangle$) readily implies the following statement.

Theorem F (Theorem 2.6.12)**.** *Let $I\subset\mathbb{C}[x,y]$ be a normal monomial ideal of finite colength. There is a bijective correspondence*

$$\left\{ \begin{array}{c} \textit{ideals } \mathfrak{n}^{\delta}_{\alpha,\beta} \textit{ appearing in the} \\ \textit{factorisation } (\triangle) \textit{ of } I \end{array} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} \textit{irreducible} \\ \textit{components of } E_I\mathbb{A}^2 \end{array} \right\},$$

where $E_I\mathbb{A}^2$ is the exceptional divisor in the blowup of $\mathbb{A}^2$ with centre the ideal I. In particular, if $J\subset\mathbb{C}[x,y]$ is an arbitrary monomial ideal and $I=\overline{J}$ is its normalisation, then $E_J\mathbb{A}^2$ has at most t irreducible components, where t is as in Equation ($\triangle$).

The Behrend number of a non-normal monomial ideal in $\mathbb{C}[x,y]$ can be computed from some explicit data defined on the normalisation of $\mathrm{Bl}_I \mathbb{A}^2$. In fact, in Section 2.7 we prove a general statement which is true in all dimensions, not just in dimension 2. We consider an *arbitrary* fat point $I\subset\mathbb{C}[x_1,\ldots,x_N]$, and the normalisation morphism

$$\mu_I\colon Z_I \to \mathrm{Bl}_I \mathbb{A}^N.$$

We let $\{D_i \mid 1\le i\le s\}$ be the irreducible components of the exceptional divisor $E_I\mathbb{A}^N\subset \mathrm{Bl}_I\mathbb{A}^N$, we set $Y_I=\mu_I^{-1}(E_I\mathbb{A}^N)$ and for each $i=1,\ldots,s$ we let $\{V_j^{(i)} \mid 1\le j\le k_i\}$ be the irreducible components of Y_I dominating D_i. We then consider the two numbers

$$d_{ij} = \deg\left(\mu_I\big|_{V_j^{(i)}}\colon V_j^{(i)} \to D_i\right), \qquad e_{ij} = \mathrm{mult}_{V_j^{(i)}}(Y_I).$$

We obtain the following result.

Theorem G (Theorem 2.7.2)**.** *Let $X\hookrightarrow\mathbb{A}^N$ be a fat point defined by an ideal $I\subset\mathbb{C}[x_1,\ldots,x_N]$. Then, there is an identity*

$$\nu_X = \sum_{i=1}^{s}\sum_{j=1}^{k_i} d_{ij}e_{ij}.$$

In Section 2.8 we argue, via an explicit example, that the toric techniques used in [31] are not directly applicable to handle fat points in higher dimensional affine spaces $\mathbb{A}^N$, for which a finer analysis is required. For instance, Theorem F fails. However, Theorem G is true in all dimensions.

Organisation of contents

After providing, in Chapter 0, some technical preliminaries and recalling some known facts, we will focus, in Chapter 1, on the study of G-constellations and, in Chapter 2, on the computation of the Behrend number of fat points.

In Chapter 0 we set up the notation, we review the notions of quotient singularities and their resolutions, fat points, cones, blowups and their normalisations.

Many of the results of this thesis have already appeared in the preprints [31, 30]. Chapter 1 mostly follows [30] and Chapter 2 mostly follows [31]. More precisely, Section 1.0 is an introduction to the theory of G-constellations and Section 1.1 is a brief introduction to the singularities of type A_{k-1} and to their crepant resolutions. Moreover, with respect to [30], a description of the partial resolutions of these singularities has been added. In Section 1.2, we will prove that the toric G-constellations are completely described in terms of certain diagrams which we will call G-stairs (Definition 1.2.19). The remaining sections of Chapter 1 contain the proofs of the main theorems of the chapter. More precisely, Theorem C is proven in Sections 1.3 and 1.4, and Theorem A is proven in Section 1.5. Finally, in Section 1.8, we will construct a crepant resolution alternative to the one constructed by Markushevich in [50]. This last construction is not present in [30].

Chapter 2 is structured as follows. In Section 2.1 we recall the definition of the Behrend function. Moreover, we prove the key result (Lemma 2.2.2) that we will exploit to perform our computations, and we compute a number of examples of Behrend functions using its elementary properties. In Section 2.3 we introduce *towers*, we completely describe their blowups (subsection 2.3.2), and we prove Theorem D. An algorithm to generalise such results is explained in Section 2.4. In Section 2.5 we prove Theorem E and Theorem F. In Section 2.7 we express the Behrend number of an arbitrary fat point $I \subset \mathbb{C}[x_1, \dots, x_N]$ in terms of data defined on the normalisation $Z_I \to \mathrm{Bl}_I \mathbb{A}^N$, thus proving Theorem G. In Section 2.8 we give an example involving fat points $X \subset \mathbb{A}^3$ showing that the analysis we carried out in $\mathbb{A}^2$ needs nontrivial modifications in order to work in higher dimension.

In Appendix A we will discuss the blowup of a smooth threefold with centre two transverse curves.

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Chapter 0

Background material

In this chapter we fix the notation used throughout the thesis, and we collect some frequently used results.

Conventions

We work over the field $\mathbb{C}$ of complex numbers. All *schemes* will be separated and of finite type over $\mathbb{C}$. A *variety* will be an integral (reduced and irreducible) scheme over $\mathbb{C}$. If X is a scheme and $V \subset X$ is a subvariety, we denote by $\mathscr{O}_{X,V}$ the local ring of X at the generic point of V. When V is an irreducible component of X, we denote by $\operatorname{mult}_V X$ the length of the local artinian ring $\mathscr{O}_{X,V}$, viewed as a module over itself. The function field of a variety X, namely the residue field of $\mathscr{O}_{X,X}$, will be denoted $\mathbb{C}(X)$. We shall denote by $\mathfrak{m} = (x, y) \subset \mathbb{C}[x, y]$ the maximal ideal of the origin $0 \in \mathbb{A}^2$.

0.1 Blowups and exceptional loci

Given a variety M along with an ideal sheaf $\mathscr{I} \subset \mathscr{O}_M$ cutting out a subscheme $X = V(\mathscr{I}) \hookrightarrow M$, we shall denote by

$$\operatorname{Bl}_{\mathscr{I}} M = \mathbb{P}_{\mathscr{O}_M}\left(\bigoplus_{i\geq 0} \mathscr{I}^i\right) \xrightarrow{\varepsilon_{\mathscr{I}}} M$$

the blowup of M along X. Sometimes we shall adopt the notation $\operatorname{Bl}_X M$, often used in the literature. The map $\varepsilon_{\mathscr{I}}$ is a projective birational (surjective) morphism of varieties which restricts to an isomorphism over $M \smallsetminus X$. The *exceptional divisor* attached to such a blowup is, by definition, the effective Cartier divisor

$$E_{\mathscr{I}} M \hookrightarrow \operatorname{Bl}_{\mathscr{I}} M \tag{0.1.1}$$

defined by the (invertible) sheaf of ideals

$$\varepsilon_{\mathscr{I}}^{-1}(\mathscr{I}) \cdot \mathscr{O}_{\operatorname{Bl}_{\mathscr{I}} M} = \operatorname{image}\left(\varepsilon_{\mathscr{I}}^{*}\mathscr{I} \to \mathscr{O}_{\operatorname{Bl}_{\mathscr{I}} M}\right).$$

In other words, $E_{\mathscr{I}} M = \mathrm{Bl}_{\mathscr{I}} M \times_M X$. If $C_{X/M} = \mathrm{Spec}_{\mathscr{O}_X}\left(\bigoplus_{i\geq 0} \mathscr{I}^i/\mathscr{I}^{i+1}\right)$ is the normal cone of the inclusion $X \hookrightarrow M$, then (0.1.1) agrees with the natural inclusion of the projective cone

$$P(C_{X/M}) = \mathbb{P}_{\mathscr{O}_X}\left(\bigoplus_{i\geq 0} \mathscr{I}^i/\mathscr{I}^{i+1}\right) = E_{\mathscr{I}} M$$

inside $\mathrm{Bl}_{\mathscr{I}} M$. Equivalently, the diagram

$$\begin{tikzcd} P(C_{X/M}) \arrow[r, hook] \arrow[d] & \mathrm{Bl}_{\mathscr{I}} M \arrow[d, "\varepsilon_{\mathscr{I}}"] \\ X \arrow[r, hook] & M \end{tikzcd}$$

is cartesian. Finally, we set

$$\mathrm{Exc}(\varepsilon_{\mathscr{I}}) = (E_{\mathscr{I}} M)_{\mathrm{red}} = P(C_{X/M})_{\mathrm{red}},$$

and we call this reduced closed subscheme of $\mathrm{Bl}_{\mathscr{I}} M$ the *exceptional locus* of the blowup. Sometimes, when no confusion is likely to arise, we shall denote it by $\mathrm{Exc}(\mathrm{Bl}_{\mathscr{I}} M)$.

Notation 0.2. More generally, given a projective birational morphism $f\colon Y \to Z$ between quasiprojective varieties, we shall denote by $\mathrm{Exc}(f) \subset Y$ the reduction of the preimage of the indeterminacy locus of the birational map f^{-1}.

We shall make extensive use of the following results.

Lemma 0.2.1 ([65, Tag 01OF])**.** *Let M be a scheme, and let $\mathscr{I}_1, \mathscr{I}_2 \subset \mathscr{O}_M$ be quasicoherent sheaves of ideals. Let $\varepsilon_{\mathscr{I}_1}\colon \mathrm{Bl}_{\mathscr{I}_1} M \to M$ be the blowup of M along $\mathscr{I}_1$. Then there is a canonical isomorphism of M-schemes*

$$\mathrm{Bl}_{\varepsilon_{\mathscr{I}_1}^{-1}(\mathscr{I}_2)\cdot\mathscr{O}_{\mathrm{Bl}_{\mathscr{I}_1} M}}(\mathrm{Bl}_{\mathscr{I}_1} M) \xrightarrow{\sim} \mathrm{Bl}_{\mathscr{I}_1\cdot\mathscr{I}_2}(M).$$

Proposition 0.2.2 ([21, Prop. IV-22])**.** *Let $M = \mathrm{Spec}\, A$ be an affine scheme, and consider a closed subscheme $X = V(f_0, f_1, \ldots, f_r) \hookrightarrow M$. The blowup of M along X agrees with the closure in $M \underset{A}{\times} \mathbb{P}^r_A = \mathbb{P}^r_A$ of the graph of the morphism*

$$\alpha_{(f_0,f_1,\ldots,f_r)}\colon M \smallsetminus X \to \mathbb{P}^r_A$$

induced by the map $\mathscr{O}_M^{\oplus(r+1)} \to \mathscr{O}_M$ sending $(a_0, a_1, \ldots, a_r) \mapsto \sum_{0\leq i\leq r} a_i f_i$.

0.3 Singularities and their resolutions

Definition 0.3.1. Let X be a quasi-projective variety. A *resolution of singularities* of X is a birational projective morphism

$$\varepsilon\colon Y \to X$$

such that Y is smooth. We will say that a resolution of singularities $\varepsilon\colon Y \to X$ is *crepant* if $\omega_Y \cong \varepsilon^*\omega_X$, where $\omega_\bullet$ denotes the canonical bundle.

Definition 0.3.2. Let X be a quasi-projective variety. A *partial resolution of singularities* is a sequence

$$Y \xrightarrow{f} Z \xrightarrow{g} X.$$

where f, g are projective and birational morphisms and $g \circ f$ is a resolution of singularities.

Notation 0.4. Sometimes we will omit the morphisms and we will refer to the varieties Y and Z respectively as resolution and partial resolution of the singularities of X.

The following classical result by Hironaka ensures the existence of the resolutions of singularities (see [46, INTRODUCTION] and reference therein).

Theorem 0.4.1 (Weak Hironaka Theorem)**.** *Let X be an irreducible reduced algebraic variety over $\mathbb{C}$ (or a suitably small neighbourhood of a compact set of an irreducible reduced analytic space) and $\mathscr{I} \subset \mathscr{O}_X$ a coherent sheaf of ideals defining a closed subscheme (or subspace) Z. Then there are a smooth variety (or analytic space) Y and a projective morphism $f : Y \to X$ such that*

- *$f^*\mathscr{I} \subset \mathscr{O}_Y$ is an invertible sheaf $\mathscr{O}_Y(-D)$ and*
- *$E_{\mathscr{I}} X + D$ is a s.n.c. divisor, i.e. its irreducible components are smooth and they intersect transversally.*

Clearly resolutions of singularities are not unique. For instance, the blowup, with a smooth centre, of a resolution produces another resolution. Although Theorem 0.4.1 guarantees the existence of resolutions of singularities, it is not true, in general, that crepant resolutions exist. Even if they exist, in dimension greater than 2, they may not be unique. Nevertheless, as mentioned in the introduction, for quotient singularities of the form $\mathbb{A}^n/G$, with $G < \mathrm{Sl}(n, \mathbb{C})$ finite and $n \leq 3$, there are crepant resolutions. For example, one is G-$\mathrm{Hilb}(\mathbb{A}^n)$ (see Chapter 1).

0.5 Finite group action on affine spaces

Given a finite group G and a representation $\rho : G \to \mathrm{GL}(n, \mathbb{C})$, we have an action of G on the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$, given by

$$\begin{aligned} G \times \mathbb{C}[x_1, \ldots, x_n] &\longrightarrow \mathbb{C}[x_1, \ldots, x_n] \\ (g, p) &\longmapsto p \circ \rho(g)^{-1} \end{aligned}$$

where p and $\rho(g)^{-1}$ are thought respectively as a polynomial and a linear function. This is equivalent to endow the structure sheaf $\mathscr{O}_{\mathbb{A}^n}$ of the affine space $\mathbb{A}^n$ with an action of the group G, which, of course, is the lift of the action on $\mathbb{A}^n$ induced by ρ. Similarly, the representation ρ induces an action of G on the tangent and cotangent sheaves and their tensor and wedge products, i.e. sheaves of the form

$$\bigwedge_{i=1}^{s} \mathscr{T}_{\mathbb{A}^n}^{\otimes m_i}$$

where $\mathscr{T}_{\mathbb{A}^n}$ is the tangent sheaf, for some $m_1, \ldots, m_s \in \mathbb{Z}$.

Out of this, we can build the quotient singularity

$$\mathbb{A}^n/G = \operatorname{Spec}\mathbb{C}[x_1,\dots,x_n]^G$$

whose points parametrize the set-theoretic orbits of the action of G on $\mathbb{A}^n$ induced by ρ.

Notice that, if the representation ρ takes values in $\mathrm{Sl}(n,\mathbb{C})$, then, the action induced by G on the canonical sheaf $\omega_{\mathbb{A}^n} = \mathscr{O}_{\mathbb{A}_n} \cdot dx_1 \wedge \cdots \wedge dx_n$ is trivial because

$$g \cdot dx_1 \wedge \cdots \wedge dx_n = \det\big(g\big)^{-1} dx_1 \wedge \cdots \wedge dx_n = dx_1 \wedge \cdots \wedge dx_n.$$

As a consequence, the singular variety $\mathbb{A}^n/G$ has trivial canonical bundle.

Given a representation $\rho : G \to \mathrm{GL}(n,\mathbb{C})$, we will say that a coherent sheaf $\mathscr{F} \in \operatorname{Ob}\operatorname{Coh}(\mathbb{A}^n)$ is *ρ-equivariant* (a *ρ-sheaf* in the sense of [9]) if there is a lift of the action $G \curvearrowright \mathbb{A}^n$ induced by ρ, i.e. for all $g \in G$ there are morphisms $\lambda_g^{\mathscr{F}} : \mathscr{F} \to \rho(g)^*\mathscr{F}$ such that:

- $\lambda_{1_G}^{\mathscr{F}} = \mathrm{id}_{\mathscr{F}}$,
- $\lambda_{hg}^{\mathscr{F}} = \rho(g)^*(\lambda_h^{\mathscr{F}}) \circ \lambda_g^{\mathscr{F}}$,

where 1_G is the unit of G. In particular, this induces a structure of representation on the vector space $H^0(\mathbb{A}^n,\mathscr{F})$ as above

$$\begin{aligned} G \times H^0(\mathbb{A}^n,\mathscr{F}) &\longrightarrow H^0(\mathbb{A}^n,\mathscr{F}) \\ (g,s) &\longmapsto (\lambda_g^{\mathscr{F}})^{-1} \circ \rho(g)^*(s). \end{aligned}$$

Whenever the representation is an inclusion $G \subset \mathrm{GL}(n,\mathbb{C})$ we will omit the representation and we will say that the sheaf is *G-equivariant* (or that it is a *G-sheaf*).

Let G be a finite group. Recall that the *regular representation* is the representation induced by the canonical (left) action of g on $\mathbb{C}[G]$, where

$$\mathbb{C}[G] = \bigoplus_{g\in G} \mathbb{C} \cdot g.$$

0.6 Fat points, monomial ideals and the Hilbert scheme

Definition 0.6.1. A *fat point* is a $\mathbb{C}$-scheme X isomorphic to $\operatorname{Spec} R$, where $(R,\mathfrak{m}_R)$ is a local artinian $\mathbb{C}$-algebra. The *embedding dimension* of a fat point $X = \operatorname{Spec} R$ is the integer $\dim_{\mathbb{C}}(\mathfrak{m}_R/\mathfrak{m}_R^2)$. When this number is 1, we say that X is *curvilinear*.

Thus a fat point is a $\mathbb{C}$-scheme X such that $X_{\mathrm{red}} \hookrightarrow X \to \operatorname{Spec}\mathbb{C}$ is the identity. In other words, it is a 0-dimensional $\mathbb{C}$-scheme whose underlying topological space is just one point. The embedding dimension of X is the smallest dimension of a smooth $\mathbb{C}$-scheme containing X as a closed subscheme.

Definition 0.6.2. The *length* of a fat point $X = \operatorname{Spec} R$ is defined as

$$\operatorname{length}(X) = \dim_{\mathbb{C}} \mathrm{H}^0(X,\mathscr{O}_X) = \dim_{\mathbb{C}}(R).$$

Notation 0.7. Occasionally, for the sake of readability, if $R = \mathbb{C}[x_1, \ldots, x_N]/I$ defines a fat point $X = \operatorname{Spec} R \subset \mathbb{A}^N$, we shall write ℓ_R instead of length(X).

Up to isomorphism of $\mathbb{C}$-schemes, there is only $\operatorname{Spec}\mathbb{C}$ of length 1, only $\operatorname{Spec}\mathbb{C}[t]/t^2$ of length 2, and only $\operatorname{Spec}\mathbb{C}[t]/t^3$ and $\operatorname{Spec}\mathbb{C}[x,y]/(x^2, xy, y^2)$ of length 3, the latter being of embedding dimension 2. If $\mathbf{k}$ is an arbitrary algebraically closed field, it is known that there is a finite number of isomorphism classes of local artinian $\mathbf{k}$-algebras of length $n \le 6$, and that this number is infinite when $n > 6$. See [58] for a complete classification of finite dimensional algebras, and [51] for a classification of $\mathbf{k}[x,y]$-*modules* of length up to 4.

Let $A = \mathbb{C}[x_1, \ldots, x_N]$ be a polynomial ring. We say that an ideal $I \subset A$ is of *finite colength* equal to n if A/I is a finite dimensional $\mathbb{C}$-vector space of dimension n, i.e. if $X = \operatorname{Spec} A/I$ is a disjoint union of fat points. Amongst all fat points $X \subset \mathbb{A}^N$ of length n, there are finitely many special ones that are cut out by *monomial* equations. There is a bijective correspondence between monomial ideals of colength n in $\mathbb{C}[x_1, \ldots, x_N]$ and $(N-1)$-dimensional partitions of n. If $N = 2$, a 1-dimensional partition corresponds to a *Ferrers diagram* (see Definition 1.2.8) (also known in the literature as a *Young tableaux* see Remark 1.2.9) made of n boxes, the correspondence being depicted in Figure 1.

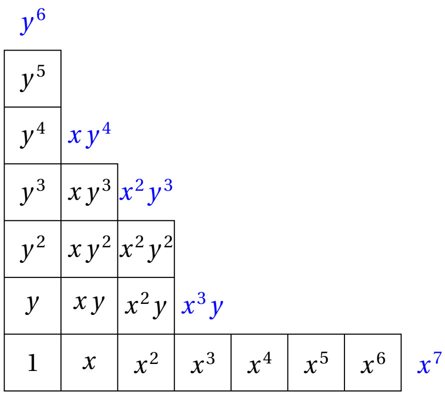


Figure 1. The Ferrers diagram corresponding to the monomial ideal $I = (x^7, x^3y, x^2y^3, xy^4, y^6)$, whose generators define the staircase of the diagram. The length (number of boxes) is 17.

Any finite subscheme $X \subset \mathbb{A}^N$ is a disjoint union of fat points. The moduli space parametrising finite subschemes $X \subset \mathbb{A}^N$ of length n is the *Hilbert scheme of points* $\operatorname{Hilb}^n(\mathbb{A}^N)$. It contains a projective subscheme $\operatorname{Hilb}^n(\mathbb{A}^N)_0 \subset \operatorname{Hilb}^n(\mathbb{A}^N)$, called the *punctual Hilbert scheme*, parametrising fat points supported at the origin $0 \in \mathbb{A}^N$. This scheme is known to be irreducible of dimension $n-1$ in the case $N = 2$, by work of Briançon [8]. It is also irreducible if $N = 3$ and $n \le 11$, by work of Jelisiejew–Keneshlou [42] while, it is reducible if $N = 3$ and $n \ge 78$ by work of Iarrobino [37]. It is remarkable that nothing about the irreducibiliy of the Hilbert scheme is known for $N = 3$ and $12 \le n \le 77$ while, (see [11]) the Hilbert scheme of points is always reducible for $N \ge 4$ and $n \le 7$ and reducible for $N \ge 4$ and $n > 7$. The locus of *all* fat points $X \subset \mathbb{A}^N$ of length n is of course given by $\mathbb{A}^N \times \operatorname{Hilb}^n(\mathbb{A}^N)_0$.

0.8 Cones

A *cone* over a scheme X is an X-scheme of the form

$$\pi\colon \operatorname{Spec}_{\mathscr{O}_X} \mathscr{A} \to X,$$

where $\mathscr{A} = \bigoplus_{i\geq 0} \mathscr{A}_i$ is a quasicoherent sheaf of graded $\mathscr{O}_X$-algebras such that the canonical map $\mathscr{A}_0 \to \mathscr{O}_X$ is an isomorphism, $\mathscr{A}_1$ is coherent and generates $\mathscr{A}$ over $\mathscr{A}_0$. Given a cone $C = \operatorname{Spec}_{\mathscr{O}_X} \mathscr{A} \to X$, one can construct another cone

$$C \oplus \mathbb{1} = \operatorname{Spec}_{\mathscr{O}_X} \mathscr{A}[z] \to X$$

where the i-th graded piece of $\mathscr{A}[z]$ is

$$(\mathscr{A}[z])_i = \mathscr{A}_i \oplus \mathscr{A}_{i-1} z \oplus \cdots \oplus \mathscr{A}_1 z^{i-1} \oplus \mathscr{A}_0 z^i.$$

On the other hand, the *projective cone* of C is defined to be the X-scheme

$$P(C) = \mathbb{P}_{\mathscr{O}_X} \mathscr{A} \to X.$$

The *projective completion* of C, namely the projective cone

$$P(C \oplus \mathbb{1}) \to X,$$

contains C as a dense open subset with closed complement $P(C) \hookrightarrow P(C \oplus \mathbb{1})$, locally cut out by the equation $z = 0$.

The main example to which these constructions apply, of crucial importance in Chapter 2, is the *normal cone* of a closed immersion $X \hookrightarrow M$ of $\mathbb{C}$-schemes, namely the cone

$$C_{X/M} = \operatorname{Spec}_{\mathscr{O}_X} \left(\bigoplus_{i\geq 0} \mathscr{I}^i / \mathscr{I}^{i+1} \right) \to X,$$

where $\mathscr{I} \subset \mathscr{O}_M$ is the ideal sheaf of $X \hookrightarrow M$.

0.9 Normalisation and order functions

Recall that a quasicompact integral scheme X is normal if for every closed point $p \in X$ the local ring $\mathscr{O}_{X,p}$ is normal (integrally closed in its field of fractions). A *normalisation* of X is a pair (Y, μ), where Y is a normal scheme and $\mu\colon Y \to X$ is a morphism such that if $\mu'\colon Y' \to X$ is a dominant morphism from a normal scheme Y', then there exists a unique morphism $\theta\colon Y' \to Y$ such that $\mu \circ \theta = \mu'$.

Proposition 0.9.1 ([49, § 4.1.2, Prop. 1.22 and 1.25])**.** *Let X be an integral scheme. Then there exists a normalisation morphism $\mu\colon Y \to X$, unique up to unique isomorphism (of X-schemes). Moreover, a morphism $f\colon Y \to X$ is the normalisation morphism if and only if Y is normal, and f is birational and integral. If X is a variety, the normalisation $\mu\colon Y \to X$ is a finite morphism.*

The following two results describe the behaviour of birational morphisms with a normal variety as a target.

Theorem 0.9.2 (Zariski's Main Theorem [35, Cor. 11.4])**.** *Let $f\colon X \to Y$ be a birational projective morphism of noetherian integral schemes, and assume that Y is normal. Then, for every point $y \in Y$, the fibre $f^{-1}(y)$ is connected.*

Lemma 0.9.3 ([65, Tag 0AB1])**.** *A finite (or even integral) birational morphism $f\colon X \to Y$ of integral schemes with Y normal is an isomorphism.*

Recall that if X is a variety and $V \hookrightarrow X$ is a prime cycle of codimension 1, the *order function* $\mathrm{ord}_V\colon \mathbb{C}(X)^\times \to \mathbb{Z}$ is defined as follows: one sets $\mathrm{ord}_V(a) = \mathrm{length}_{\mathscr{O}_{X,V}}(\mathscr{O}_{X,V}/a\cdot\mathscr{O}_{X,V})$ for $a \in \mathscr{O}_{X,V}$, and for $h = a/b \in \mathbb{C}(X)^\times$, one proves easily that the definition $\mathrm{ord}_V(h) = \mathrm{ord}_V(a) - \mathrm{ord}_V(b)$ is well given. This definition generalises the more familiar notion that applies when X is normal: in this case, the local ring $(\mathscr{O}_{X,V}, \mathfrak{m}_{X,V})$ is a discrete valuation ring (and not just a local integral domain), and one defines $\mathrm{ord}_V(a)$ to be the largest integer k such that $a \in \mathfrak{m}_{X,V}^k$. The two notions are related as shown in the next result.

Proposition 0.9.4 ([26, Ex. 1.2.3])**.** *Let X be a variety, $\mu\colon Y \to X$ the normalisation of X, and let $V \hookrightarrow X$ be a subvariety of codimension 1. If $h \in \mathbb{C}(X)^\times = \mathbb{C}(Y)^\times$, then*

$$\mathrm{ord}_V(h) = \sum_{\mu\colon W \to V} \mathrm{ord}_W(h)\cdot[\mathbb{C}(W):\mathbb{C}(V)],$$

where the sum is over all subvarieties $W \hookrightarrow Y$ which map onto V, and $[\mathbb{C}(W):\mathbb{C}(V)]$ denotes the degree of the corresponding field extension.

0.10 Normalisation of blowups

Recall that if I is an ideal in a polynomial ring $A = \mathbb{C}[x_1,\ldots,x_N]$, then the *Rees algebra* of I is

$$A[It] = A \oplus It \oplus I^2t^2 \oplus I^3t^3 \oplus \cdots \subset A[t].$$

Since A is a domain, so is $A[It]$, see [65, Tag 01OF], and therefore the blowup $\mathrm{Bl}_I \mathbb{A}^N$ is again a variety (an integral scheme of finite type over $\mathbb{C}$). If I is monomial, by the general theory of normalisation in this setup (see e.g. [48, § II.5] for a thorough treatment), the integral closure of $A[It]$ is

$$\overline{A[It]} = A \oplus \overline{I}t \oplus \overline{I^2}t^2 \oplus \overline{I^3}t^3 \oplus \cdots$$

where, after setting $x^m = x_1^{m_1}\cdots x_N^{m_N}$ for $m = (m_1,\ldots,m_N) \in \mathbb{N}^N$, one defines

$$\overline{I^i} = \left(x^m \in A \,\middle|\, (x^m)^p \in I^{ip} \text{ for some } p \geq 1\right) \subset A. \tag{0.10.1}$$

By [57, Ex. 6.C.9], the inclusion $A[It] \hookrightarrow \overline{A[It]}$ induces an everywhere defined morphism

$$\mu_I\colon \mathbb{P}\overline{A[It]} \to \mathrm{Bl}_I \mathbb{A}^N,$$

which agrees with the normalisation morphism. In general, $A[It]$ is normal if and only if $I^i = \overline{I^i}$ for every $i \geq 1$.

By [59, Prop. 3.1], in the case $N = 2$, the algebra $\overline{\mathbb{C}[x,y][It]}$ agrees with the Rees algebra $\mathbb{C}[x,y][\overline{I}t]$ of the monomial ideal $\overline{I} \subset \mathbb{C}[x,y]$. In particular the normalisation of $\mathrm{Bl}_I \mathbb{A}^2$ is the blowup of $\mathbb{A}^2$ along $\overline{I}$. This motivates the following common terminology.

Definition 0.10.1. We will say that an ideal $I \subset A$ is *normal* if its Rees algebra $A[It]$ is normal. When $A = \mathbb{C}[x,y]$, we will call $\overline{I}$ the *normalisation* of I.

We next state a special case of [17, Prop. 1.1] suited for our purposes (the general statement involves a polynomial ring in an arbitrary number of variables).

Proposition 0.10.2 ([17, Prop. 1.1])**.** *Let* $I = (x^{a_1}, x^{a_2}y^{b_2}, \ldots, x^{a_{s-1}}y^{b_{s-1}}, y^{b_s}) \subset \mathbb{C}[x,y]$ *be a monomial ideal of finite colength and let* $Q_I \subset \mathbb{R}^2$ *be the subset defined by*

$$Q_I = \operatorname{Conv}_{\mathbb{Q}}((a_1,0),(a_2,b_2),\ldots,(a_{s-1},b_{s-1}),(0,b_s)) + \mathbb{Q}^2_{\geq 0}$$

where $\operatorname{Conv}_{\mathbb{Q}}(p_1,\ldots,p_s) \subset \mathbb{Q}^2$ *denotes the convex hull of a set of points* $p_1,\ldots,p_s \in \mathbb{N}^2 \subset \mathbb{Q}^2$*. Then, for* $i \neq 0$*, one has*

$$\overline{I^i} = \left(x^a y^b \,\middle|\, (a,b) \in i \cdot Q_I \cap \mathbb{Z}^2 \right).$$

Remark 0.10.3. Proposition 0.10.2 provides a criterion to establish whether, given a monomial ideal of finite colength $I \subset \mathbb{C}[x,y]$, the blowup variety $\operatorname{Bl}_I \mathbb{A}^2$ is normal or not. Explicitly, if Q_I is defined as in Proposition 0.10.2 and $A_I = \{(a,b) \in \mathbb{N}^2 \mid x^a y^b \in I\}$, then $\operatorname{Bl}_I \mathbb{A}^2$ is normal if and only if

$$A_I = Q_I \cap \mathbb{N}^2.$$

Moreover, we have the equality $Q_I = \operatorname{Conv}_{\mathbb{Q}}(A_I)$ and, as a consequence, if $I, J \subset \mathbb{C}[x,y]$ are normal ideals, then IJ is normal. Indeed, by general properties of convexes (see [13, § 2.2.]), we have

$$\begin{aligned} Q_{IJ} \cap \mathbb{N}^2 &= \operatorname{Conv}_{\mathbb{Q}}(A_{IJ}) \cap \mathbb{N}^2 \\ &= \operatorname{Conv}_{\mathbb{Q}}(A_I + A_J) \cap \mathbb{N}^2 \\ &= (\operatorname{Conv}_{\mathbb{Q}}(A_I) + \operatorname{Conv}_{\mathbb{Q}}(A_J)) \cap \mathbb{N}^2 \\ &= A_I + A_J \\ &= A_{IJ}. \end{aligned}$$

Notice that the converse in not true. For instance, setting $\mathfrak{m} = (x,y)$, we have

$$\mathfrak{m}^3 = \mathfrak{m} \cdot (x^2, y^2),$$

and we shall see in Example 0.10.8 that (x^2, y^2) is not normal.

Example 0.10.4. Consider the two ideals $I = (x^2, y^2)$ and $J = (x^2, y^3)$ in $\mathbb{C}[x,y]$. Then,

$$\begin{aligned} A_I &= \left\{ (a,b) \in \mathbb{N}^2 \,\middle|\, a, b \geq 2 \right\}, \\ A_J &= \left\{ (a,b) \in \mathbb{N}^2 \,\middle|\, a \geq 2, b \geq 3 \right\}. \end{aligned}$$

Since $(1,1) \in (Q_I \cap \mathbb{N}^2) \smallsetminus A_I$ and $(1,2) \in (Q_J \cap \mathbb{N}^2) \smallsetminus A_J$, the blowups $\operatorname{Bl}_I \mathbb{A}^2$ and $\operatorname{Bl}_J \mathbb{A}^2$ are not normal. The integral closures of the Rees algebras are respectively given by

$$\begin{aligned} \overline{\mathbb{C}[x,y][It]} &= \mathbb{C}[x,y][\overline{I}t] \\ \overline{\mathbb{C}[x,y][Jt]} &= \mathbb{C}[x,y][\overline{J}t] \end{aligned}$$

where $\overline{I} = \mathfrak{m}^2$ and $\overline{J} = (x^2, xy^2, y^3)$.

We will see many examples of normalisations of blouwps of the affine plane $\mathbb{A}^2$ with centre a monomial ideal of finite colength $I \subset \mathbb{C}[x,y]$ in Sections 2.5 and 2.7.

Ferrers diagrams of ideals which have normal Rees algebras admit a useful description that was given in [28] and we present below.

Theorem 0.10.5 ([28, Thm. 2.13])**.** *Let $I \subset \mathbb{C}[x,y]$ be minimally generated by the $n+1$ monomials*

$$x^{a_0}, x^{a_1}y^{b_{n-1}}, \dots, x^{a_i}y^{b_{n-i}}, \dots, x^{a_{n-1}}y^{b_1}, y^{b_0} \tag{0.10.2}$$

where $a_i > a_{i+1}$ and $b_i > b_{i+1}$ for $i = 0, \dots, n-2$. Set $a_n = b_n = 0$. If the blowup $\mathrm{Bl}_I \mathbb{A}^2$ is normal, then there exists an integer k such that $0 \le k \le n$ and

1. $a_n = 0, a_{n-1} = 1, a_{n-2} = 2, \dots, a_k = n-k$,
2. $b_n = 0, b_{n-1} = 1, b_{n-2} = 2, \dots, b_{n-k} = k$,
3. $b_i \le \lceil \frac{b_{i-1}+b_{i+1}}{2} \rceil$ *for* $i = 1, \dots, n-k-1$,
4. $a_i \le \lceil \frac{a_{i-1}+a_{i+1}}{2} \rceil$ *for* $i = 1, \dots, k-1$.

Remark 0.10.6. If an ideal $I \subset \mathbb{C}[x,y]$, as in (0.10.2), is normal, then the boundary ∂Q_I of Q_I is piece-wise linear, i.e.

$$\partial Q_I = \overline{\mathbb{Q}^2 \smallsetminus Q_I} \cap Q_I = s_0 \cup \dots \cup s_{t+1},$$

where $\overline{\mathbb{Q}^2 \smallsetminus Q_I}$ denotes the Euclidean closure and $s_0, \dots, s_{t+1}$ are, possibly unbounded, segments with different slopes. Let us also denote by $v_0, \dots, v_t$ the vertices of ∂Q_I i.e.

$$\{\, v_i \mid 0 \le i \le t \,\} = \{\, s_i \cap s_j \mid 0 \le i < j \le t+1 \,\}.$$

For instance, $I = (1)$ if and only if $t = 0$ which also implies $v_0 = (0,0)$.

Then, as a consequence of Theorem 0.10.5, up to relabeling the linear pieces and the vertices, the following properties hold:

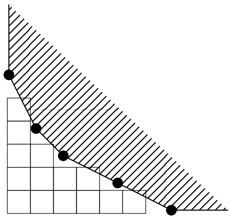
- the pieces s_0 and s_{t+1} are unbounded and respectively supported on the positive half horizontal axis and on the positive half vertical axis,
- for $i = 0, \dots, t$, we have $v_i = s_i \cap s_{i+1}$,
- for $i = 1, \dots, t$, the segments s_i are bounded and supported on certain lines $r_1, \dots, r_t$, such that each line r_i has negative slope m_i and $0 > m_i > m_{i+1}$ for all $i = 1, \dots, t-1$,
- there is a strictly increasing sequence $0 = k_0 < \dots < k_t = n$ of positive integers such that $v_i = (a_{k_i}, b_{n-k_i})$, where, as above, we set $a_n = b_n = 0$.

Now, the integer k of Theorem 0.10.5, can be chosen as

$$k = \max\{\, i \mid m_i \ge -1 \,\}.$$

Below we show an example of Ferrers diagram of a monomial 0-dimensional scheme whose associated Rees algebra is normal.

Example 0.10.7. Consider the ideal $I = (x^6, x^4y, x^2y^2, xy^3, y^5) \subset \mathbb{C}[x, y]$ and Q_I, A_I as in Proposition 0.10.2 and Remark 0.10.3. The Ferrers diagram of I is



where the highlighted area in the above picture corresponds to Q_I. Then, the ideal I is normal because $A_I = Q_I \cap \mathbb{N}^2$.

Example 0.10.8. Set $I = (x^k, y^k) \subset \mathbb{C}[x, y]$, where $k > 1$. Then I is not normal, and the normalisation of $\mathrm{Bl}_I \mathbb{A}^2$ is given by $\mathrm{Bl}_{\mathfrak{m}^k} \mathbb{A}^2$. Moreover, as we shall see in Example 2.2.7 (but see also [35, Ex. II.7.11]), there is a canonical isomorphism

$$\mathrm{Bl}_{\mathfrak{m}} \mathbb{A}^2 \xrightarrow{\sim} \mathrm{Bl}_{\mathfrak{m}^k} \mathbb{A}^2 = \mathrm{Bl}_{\overline{I}} \mathbb{A}^2.$$

Composing with the normalisation morphism, one obtains a morphism

$$\mathrm{Bl}_{\mathfrak{m}} \mathbb{A}^2 \to \mathrm{Bl}_I \mathbb{A}^2,$$

induced of course by $I \subset \overline{I} \subset \mathfrak{m}$. An example with $k = 5$ is depicted inFigure 2.

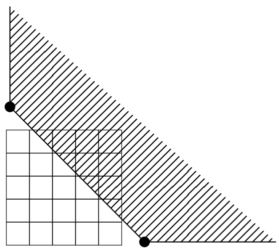


Figure 2. The normalisation of the ideal $I = (x^5, y^5)$ is $\mathfrak{m}^5$.

Again, as in Example 0.10.7, if $Q_I \subset \mathbb{Q}^2$ is defined as in Proposition 0.10.2, then the highlighted area in the above figure corresponds to Q_I, but this time $Q_I \cap \mathbb{N}^2 \neq A_I$.

Example 0.10.9. In general, if $I = (x^k, y^h)$, the blowup $\mathrm{Bl}_I \mathbb{A}^2$ is canonically isomorphic (see [21, Prop. IV-25]), over $\mathbb{A}^2$, to the quasiprojective surface

$$B = \left\{ ((x, y), [u : v]) \in \mathbb{A}^2 \times \mathbb{P}^1 \,\middle|\, vx^k = uy^h \right\} \hookrightarrow \mathbb{A}^2 \times \mathbb{P}^1.$$

In particular, for $h, k > 1$, the blowup $\mathrm{Bl}_I \mathbb{A}^2$ is singular along the exceptional divisor and hence, it is not normal.

0.11 Self-intersection inside quasiprojective surfaces

Let $B = \mathrm{Bl}_I \mathbb{A}^2$ be the blowup of the affine plane with centre a fat point supported at the origin. Then B contains a finite number of irreducible projective rational curves $C_1, \ldots, C_r$.

Suppose that B is smooth. Whenever we will talk about self-intersection we will refer to the quadratic form on $\bigoplus_{1\leq i\leq r}\mathbb{Z}C_i$ induced by the map

$$\begin{aligned} \{C_1,\dots,C_r\} &\xrightarrow{(-)^2} \mathbb{Z} \\ C_i &\longmapsto C_i^2 = \deg_{C_i}\left(\mathscr{O}_B(C_i)\big|_{C_i}\right). \end{aligned}$$

Chapter 1

Moduli spaces of $\mathbb{Z}/k\mathbb{Z}$-constellations over $\mathbb{A}^2$

1.0 G-clusters & G-constellations

Definition 1.0.1. Let $G \subset \mathrm{GL}(n,\mathbb{C})$ be a finite subgroup. A G*-cluster* is a 0-dimensional subscheme Z of $\mathbb{A}^n$ such that:

- the structure sheaf $\mathcal{O}_Z$ is G-equivariant, i.e. the ideal I_Z is invariant with respect to the action of G on $\mathbb{C}[x_1,\dots,x_n]$, and
- if $\rho_{\mathrm{reg}} : G \to \mathrm{GL}(\mathbb{C}[G])$ is the regular representation, then there is an isomorphism of representations
$$\varphi : H^0(Z,\mathcal{O}_Z) \to \mathbb{C}[G],$$
i.e. φ is an isomorphism of vector spaces such that the following diagram:
$$\begin{tikzcd} G \times H^0(Z,\mathcal{O}_Z) \arrow[r] \arrow[d, "\mathrm{id}_G \times \varphi"'] & H^0(Z,\mathcal{O}_Z) \arrow[d, "\varphi"] \\ G \times \mathbb{C}[G] \arrow[r] & \mathbb{C}[G] \end{tikzcd}$$
where the horizontal arrows are the G-actions, commutes.

Example 1.0.2. Consider the representation
$$\begin{aligned} \rho_1 :\ \mathbb{Z}/3\mathbb{Z} &\longrightarrow \mathrm{GL}(2,\mathbb{C}) \\ 1 &\longmapsto \begin{pmatrix} \xi_3 & 0 \\ 0 & \xi_3^2 \end{pmatrix}. \end{aligned}$$
where ξ_3 is a primitive third root of 1. Then, the following are examples of $\mathbb{Z}/3\mathbb{Z}$-clusters of $\mathbb{A}^2$:
$$\mathrm{Spec}\left(\frac{\mathbb{C}[x,y]}{(x,y^3)}\right),\ \mathrm{Spec}\left(\frac{\mathbb{C}[x,y]}{(x,y)^2}\right),\ \mathrm{Spec}\left(\frac{\mathbb{C}[x,y]}{(x-a,y-b)(x-\xi a,y-\xi^2 b)(x-\xi^2 a,y-\xi b)}\right)$$
for some $(a,b) \in \mathbb{A}^2 \setminus 0$.

Remark 1.0.3. In general, a free orbit is always a G-cluster. On the other side, the support of a G-cluster, i.e. its reduction, is always a union of orbits of the G-action. We will see that the G-cluster whose support is one orbit will play a central role in the theory developed.

Definition 1.0.4. We will denote by $\mathrm{Hilb}^G(\mathbb{A}^n)$ the fine moduli space of G-clusters and, by G-$\mathrm{Hilb}(\mathbb{A}^n)$ the irreducible component of $\mathrm{Hilb}^G(\mathbb{A}^n)$ containing the free G-orbits.

Remark 1.0.5. The scheme Hilb-$G(\mathbb{A}^n)$ can be constructed as a closed subscheme of the Hilbert scheme of $|G|$ points $\mathrm{Hilb}^{|G|}(\mathbb{A}^n)$, i.e. the fine moduli space of length $|G|$ subschemes of the affine space. Its existence and the fact that it is quasi-projective was proven in [34].

Theorem 1.0.6 ([9, Theorem 1.2])**.** *Let $G \subset \mathrm{Sl}(n,\mathbb{C})$ be a finite subgroup where $n = 2,3$. Then $\mathbb{A}^n/G$ has only Gorenstein singularities. Moreover the Hilbert–Chow morphism*
$$Y := G\text{-}\mathrm{Hilb}(\mathbb{A}^n) \xrightarrow{\varepsilon} \mathbb{A}^n/G =: X,$$
which associates to each G-cluster its support, is a crepant resolution of singularities, i.e. $\omega_Y \cong \varepsilon^*\omega_X$. (see Definition 0.3.1)

Remark 1.0.7. The Hilbert–Chow morphism ε mentioned in Theorem 1.0.6 is a G-equivariant version of the usual Hilbert–Chow morphism

$$\overline{\varepsilon} : \mathrm{Hilb}^{|G|}(\mathbb{A}^n) \to \mathrm{Sym}^{|G|}(\mathbb{A}^n),$$

which associates to each 0-dimensional subscheme $Z \subset \mathbb{A}^n$ of length n its support $\mathrm{Supp}(Z)$. In particular ε can be thought of as the restriction of $\overline{\varepsilon}$ to the G-invariant subvariety $G\text{-Hilb}(\mathbb{A}^n) \subset \mathrm{Hilb}^{|G|}(\mathbb{A}^n)$.

Notice that the existence of the Hilbert–Chow morphism guarantees that $G\text{-Hilb}(\mathbb{A}^n)$ contains, for $n = 2, 3$, only (see Remark 1.0.3). G-cluster whose support is one G-orbit.

A natural generalisation of the concept of G-cluster is given in [15], and it is achieved by consider coherent $\mathscr{O}_{\mathbb{A}^n}$ -modules which are not necessarily the structure sheaves of 0-dimensional subschemes of $\mathbb{A}^n$.

Definition 1.0.8 ([15, Definition 2.1])**.** Let $G \subset \mathrm{GL}(n, \mathbb{C})$ be a finite subgroup. A G*-constellation* is a coherent $\mathscr{O}_{\mathbb{A}^n}$-module $\mathscr{F}$ on $\mathbb{A}^n$ such that:

- $\mathscr{F}$ is G-equivariant, and
- there is an isomorphism of representations

$$\varphi : H^0(\mathbb{A}^n, \mathscr{F}) \to \mathbb{C}[G].$$

Remark 1.0.9. Since a G-constellation $\mathscr{F}$ is a coherent sheaf on the affine variety $\mathbb{A}^n$, sometimes, by abuse of notations, we will call G-constellation its global sections $H^0(\mathscr{F}, \mathbb{A}^n)$ as well as $\mathscr{F}$ and, sometimes, we will treat a G-constellation as if it were a $\mathbb{C}[x_1, \ldots, x_n]$-module, meaning that we are working with the space of its global sections.

Definition 1.0.10. A G-constellation $\mathscr{F}$ is *irreducible* if it cannot be written as a direct sum

$$\mathscr{F} = \mathscr{E}_1 \oplus \mathscr{E}_2,$$

where $\mathscr{E}_1, \mathscr{E}_2$ are proper G-subsheaves, and it is *reducible* otherwise.

Example 1.0.11. The structure sheaf of a G-cluster is a G-constellation. In particular, a G-cluster is irreducible if and only its support is one G-orbit (see Remark 1.0.3).

Remark 1.0.12. If we think of a G-constellation as its global sections, a G-constellation $F = H^0(\mathscr{F}, \mathbb{A}^n)$ is irreducible if it cannot be written as a direct sum

$$F = E_1 \oplus E_2,$$

where E_1, E_2 are proper G-equivariant $\mathbb{C}[x_1, \ldots, x_n]$-submodules.

Remark 1.0.13. The G-equivariance hypothesis implies that the support of a G-constellation $\mathscr{F}$ is a union of G-orbits. When $\mathscr{F}$ is irreducible, then its support must be a G-orbit. Moreover, for dimensional reasons, the only constellations supported on a free orbit Z are isomorphic to the structure sheaf $\mathscr{O}_Z$.

Remark 1.0.14. Recall that (see, for example, [27, chapters 1 and 2]), given a finite group G and the set of isomorphism classes of its irreducible representations

$$\mathrm{Irr}(G) = \{\text{Irreducible representations}\}/\mathrm{iso},$$

there is a ring isomorphism

$$\Psi : R(G) \xrightarrow{\sim} \bigoplus_{\rho \in \mathrm{Irr}(G)} \mathbb{Z}\rho$$

where $(R(G), \oplus)$ is the Grothendieck group of isomorphism classes of representations of G, and the ring structure (on both sides) is induced by tensor product $\otimes$ of representations. Moreover $\mathrm{Irr}(G) = \{\rho_1, \dots, \rho_s\}$ is finite, and we have the correspondence:

$$\begin{array}{ccc} R(G) & \xrightarrow{\quad\Psi\quad} & \bigoplus_{i=1}^{s} \mathbb{Z}\rho_i \\ \mathbb{C}[G] & \longmapsto & (\dim\rho_1, \dots, \dim\rho_s). \end{array}$$

Following the ideas in [45], the above mentioned properties allow one to introduce a notion of stability on the set of G-constellations. Given a finite subgroup $G \subset \mathrm{Sl}(n, \mathbb{C})$ (where $n = 2, 3$), the *space of stability conditions* for G-constellations is

$$\Theta = \{\, \theta \in \mathrm{Hom}_{\mathbb{Z}}(R(G), \mathbb{Q}) \mid \theta(\mathbb{C}[G]) = 0 \,\}$$

Definition 1.0.15. Let $\theta \in \Theta$ be a stability condition. A G-constellation $\mathscr{F}$ is said to be θ-*(semi)stable* if, for any proper G-equivariant subsheaf $0 \subsetneq \mathscr{E} \subsetneq \mathscr{F}$, we have

$$\theta(H^0(\mathbb{A}^n, \mathscr{E})) \underset{(\geq)}{>} 0.$$

A stability condition θ is *generic* if the notion of θ-semistability is equivalent to the notion of θ-stability. Finally, we will denote by $\Theta^{\mathrm{gen}} \subset \Theta$ the subset of generic stability conditions.

Remark 1.0.16. If $\mathscr{F}$ is reducible, then it is not θ-stable for any generic stability condition $\theta \in \Theta^{\mathrm{gen}}$. Indeed, suppose that $\mathscr{F} = \mathscr{E}_1 \oplus \mathscr{E}_2$ for some proper G-subsheaves $\mathscr{E}_1, \mathscr{E}_2 \subsetneq \mathscr{F}$. Then, if $\mathscr{F}$ had been θ-stable for some $\theta \in \Theta^{\mathrm{gen}}$, we would have

$$\begin{cases} 0 = \theta(\mathscr{F}) = \theta(\mathscr{E}_1) + \theta(\mathscr{E}_2), \\ \theta(\mathscr{E}_1) > 0, \\ \theta(\mathscr{E}_2) > 0, \end{cases}$$

which yields a contradiction.

Since, for our purpose, we will be interested in irreducible G-constellations, whenever not specified a G-constellation will always be irreducible.

Remark 1.0.17. If $Z \subset \mathbb{A}^n$ is a free orbit, then $\mathscr{O}_Z$ does not admit any proper G-subsheaf. Indeed, given a nonzero element $s \in \mathscr{O}_Z$, the collection $\{\, g \cdot s \mid g \in G \,\}$ generates $\mathscr{O}_Z$. As a consequence, $\mathscr{O}_Z$ is θ-stable for all $\theta \in \Theta$.

Example 1.0.18. Let us adopt the notation in Example 1.0.2. Then, the $\mathbb{C}[x,y]$-modules

$$F_1 = \frac{(x,y)}{(x^2,y^2)},\ F_2 = \frac{(x)}{(x,y)^3},\ F_3 = \frac{(x,y)^2}{(x,y)^3}$$

define three $\mathbb{Z}/3\mathbb{Z}$-constellations. Their $\mathbb{Z}/3\mathbb{Z}$-equivariant proper submodules are

	G-submodules	$R(G)$
F_1	$\frac{(x)}{(x^2,y^2)}$	$(1,1,0)$
	$\frac{(y)}{(x^2,y^2)}$	$(1,0,1)$
	$\frac{(xy)}{(x^2,y^2)}$	$(1,0,0)$
F_2	$\frac{(x^2)}{(x,y)^3}$	$(0,0,1)$
	$\frac{(xy)}{(x,y)^3}$	$(1,0,0)$
	$\frac{(x^2,xy)}{(x,y)^3}$	$(1,0,1)$
F_3	$\frac{(x^2)}{(x,y)^3}$	$(0,0,1)$
	$\frac{(xy)}{(x,y)^3}$	$(1,0,0)$
	$\frac{(y^2)}{(x,y)^3}$	$(0,1,0)$

In the last column are listed the correspondent elements in $R(G)$ (see Remark 1.0.13). As a consequence, if $\theta_1 = (2,-1,-1)$ and $\theta_1 = (1,-2,1)$ are two stability conditions, then F_1 (resp. F_2) is θ_1-stable (resp. θ_2-stable) and it is not θ_2-stable (resp. θ_1-stable). Finally, F_3, which is reducible, is not θ_i-stable for $i = 1,2$. One can also show that both θ_1 and θ_2 are generic conditions.

Definition 1.0.19. Let $\theta \in \Theta^{\text{gen}}$ be a generic stability condition. We call $\mathcal{M}_\theta$ the irreducible component of the *(fine) moduli space of θ-stable G-constellations* containing the free orbits.

In this context, we will denote by $\mathcal{U}_\theta$ the universal family of θ-stable G-constellations, namely $\mathcal{U}_\theta \in \operatorname{Ob}\operatorname{Coh}(\mathcal{M}_\theta \times \mathbb{A}^n)$, and by $\mathcal{R}_\theta$ the tautological bundle $\mathcal{R}_\theta := (\pi_{\mathcal{M}_\theta})_* \mathcal{U}_\theta$.

Remark 1.0.20. The tautological bundle $\mathcal{R}_\theta$ is the vector bundle of rank $|G|$ whose fibre, over the point $[\mathcal{F}] \in \mathcal{M}_\theta$, is the complex vector space $H^0(\mathbb{A}^2,\mathcal{F})$.

The theorem below brings together results from [45, 15, 9].

Theorem 1.0.21. *The following results are true for $n = 2,3$.*

- *The subset $\Theta^{\text{gen}} \subset \Theta$ of generic parameters is open and dense. It is the disjoint union of finitely many open convex polyhedral cones in Θ called chambers.*

- *For generic $\theta \in \Theta^{\text{gen}}$, the moduli space $\mathcal{M}_\theta$ exists and it depends only upon the chamber $C \subset \Theta^{\text{gen}}$ containing θ, so we write $\mathcal{M}_C$, $\mathcal{U}_C$ and $\mathcal{R}_C$ in place of $\mathcal{M}_\theta$, $\mathcal{U}_\theta$, and $\mathcal{R}_\theta$ for any*

$\theta \in C$. Moreover, the Hilbert–Chow morphism, which associates to each G-constellation $\mathcal{F}$ its support $\mathrm{Supp}(\mathcal{F})$*, $\varepsilon\colon \mathcal{M}_C \to \mathbb{A}^n/G$, is a crepant resolution.*

- *(Craw–Ishii Theorem) Given a finite abelian subgroup $G \subset \mathrm{SL}(n,\mathbb{C})$, suppose $Y \xrightarrow{\varepsilon} \mathbb{A}^n/G$ is a projective crepant resolution. Then $Y \cong \mathcal{M}_C$ for some chamber $C \subset \Theta$ and $\varepsilon = \varepsilon_C$ is the Hilbert-Chow morphism.*
- *There exists a chamber $C_G \subset \Theta^{\mathrm{gen}}$ such that $\mathcal{M}_{C_G} = G\text{-}\mathrm{Hilb}(\mathbb{A}^n)$.*

Remark 1.0.22. As expected by Theorem 1.0.6 and Remarks 1.0.3 and 1.0.16, $\mathcal{M}_{C_G} \cong G$-Hilb$(\mathbb{A}^n)$ parametrises irreducible G-clusters.

Remark 1.0.23. Let $U_C = \mathcal{M}_C \smallsetminus \mathrm{Exc}(\varepsilon_C)$ be the complement of the exceptional locus of the Hilbert-Chow morphism. Then, Remarks 1.0.13 and 1.0.17 imply, together with the third point of Theorem 1.0.21, that for any two chambers $C, C' \subset \Theta^{\mathrm{gen}}$ we have a canonical isomorphism of families over $\mathbb{A}^n/G$-schemes

$$\begin{array}{ccc} \mathcal{U}_C|_{U_C\times\mathbb{A}^n} & & \mathcal{U}_{C'}|_{U_{C'}\times\mathbb{A}^n} \\ \Big| & \cong & \Big| \\ U_C\times\mathbb{A}^n & & U_{C'}\times\mathbb{A}^n, \end{array}$$

i.e. there exists a unique isomorphism $\varphi_C : U_C \to U_{C'}$ such that the diagram

$$\begin{tikzcd} U_C \arrow[rr, "\varphi"] \arrow[dr, "\varepsilon_C"'] & & U_{C'} \arrow[dl, "\varepsilon_{C'}"] \\ & \mathbb{A}^n/G & \end{tikzcd}$$

commutes and $\mathcal{U}_C|_{U_C\times\mathbb{A}^n} \cong (\varphi\times \mathrm{id}_{\mathbb{A}^n})^*\mathcal{U}_{C'}|_{U_{C'}\times\mathbb{A}^n}$.

In particular, for any C, U_C parametrizes the free orbits of the G-action as the complement of the singular locus of $\mathbb{A}^n/G$ does.

We conclude this section with the statement of the Craw-Ishii conjecture.

Conjecture 1.0.24. *Let $Y \to X$ be a crepant resolution of a quotient singularity $X = \mathbb{A}^3/G$, for $G < \mathrm{Sl}(3,\mathbb{C})$ finite. Then, there exists a generic stability condition $\theta \in \Theta^{\mathrm{gen}}$ such that $\mathcal{M}_\theta$ and Y are isomorphic over X.*

1.1 The two-dimensional abelian case

In this section we will fix some notation that we will use throughout the rest of this chapter and will give a very brief description of the singularities of type $A_{|G|-1}$ and of their respective resolutions. Moreover, we will give an explicit construction of its partial resolutions as blowups with centre some nonreduced ideal.

Throughout the section, we will consider a finite abelian subgroup $G \subset \mathrm{SL}(n,\mathbb{C})$.

1.1.1 The action of G

Whenever $G \subset \mathrm{SL}(n,\mathbb{C})$ is a finite abelian subgroup, it is well known that its irreducible representations are 1-dimensional and that there is a bijection between the group G and $\mathrm{Irr}(G)$. Moreover, the map Ψ in Remark 1.0.14 is such that

$$\begin{aligned} R(G) &\xrightarrow{\ \Psi\ } \bigoplus_{\rho\in\mathrm{Irr}(G)} \mathbb{Z}\rho \\ \mathbb{C}[G] &\longmapsto (1,\dots,1). \end{aligned}$$

In particular, in dimension 2, it is well known that all finite abelian subgroups $G \subset \mathrm{Sl}(2,\mathbb{C})$ are cyclic. Moreover, for any $k \geq 1$, there is only one conjugacy class of abelian subgroups of $\mathrm{SL}(2,\mathbb{C})$ isomorphic to $\mathbb{Z}/k\mathbb{Z}$. In what follows we will choose, as representative of such conjugacy class,

$$\mathbb{Z}/k\mathbb{Z} \cong G = \langle g_k = \begin{pmatrix} \xi_k^{-1} & 0 \\ 0 & \xi_k \end{pmatrix} \rangle \subset \mathrm{SL}(2,\mathbb{C}) \tag{1.1.1}$$

where ξ_k is a (fixed) primitive k-th root of unity.

We will adopt the following notation for the irreducible representations of G:

$$\mathrm{Irr}(G) = \left\{ \begin{array}{rl} \rho_i: \mathbb{Z}/k\mathbb{Z} & \longrightarrow \mathbb{C}^* \\ g_k & \longmapsto \xi_k^i \end{array} \,\middle|\, i = 0,\dots,k-1 \right\}. \tag{1.1.2}$$

Sometimes, we will identify $\mathrm{Irr}(G)$ with the set $\{0,\dots,k-1\}$ according to the bijection $\rho_j \mapsto j$. Notice that one may also identify $(\mathrm{Irr}(G),\otimes)$ with the abelian group $(\mathbb{Z}/k\mathbb{Z},+)$, but, in what follows, we will mostly deal with $\mathrm{Irr}(G)$ as a set of indices, hence we will ignore the natural group structure on it.

1.1.2 The quotient singularity $\mathbb{A}^2/G$ and its resolution

The singularity obtained in this case is called A_{k-1} (or *Kleinian* or *DuVal*) singularity, i.e.

$$A_{k-1} := \mathbb{A}^2/G.$$

This is a rational double point and it has been intensively studied for decades (see for instance [19]). It is well known that it has a unique minimal, in fact crepant, resolution $Y \xrightarrow{\varepsilon} A_{k-1}$ whose exceptional divisor is a chain of $k-1$ smooth (-2)-rational projective curves.

As a consequence of Theorem 1.0.21 and of the uniqueness of the minimal model of a surface, for any chamber C, there is an isomorphism of varieties $\varphi_C : \mathscr{M}_C \xrightarrow{\sim} Y$ such that the diagram

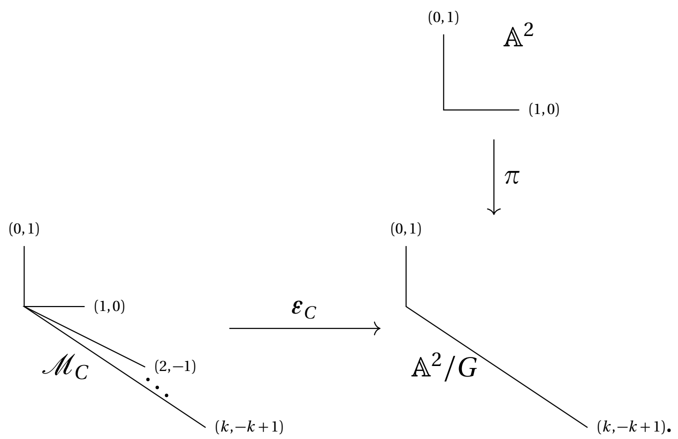
$$\begin{tikzcd} \mathscr{M}_C \arrow[rr, "\varphi_C"] \arrow[dr, "\varepsilon_C"'] & & Y \arrow[dl, "\varepsilon"] \\ & A_{k-1} & \end{tikzcd}$$

commutes. What changes between two different chambers C, C' is that they have different universal families $\mathscr{U}_C, \mathscr{U}_{C'} \in \operatorname{Ob}\operatorname{Coh}(Y \times \mathbb{A}^2)$.

Let us describe in details the minimal resolution and the partial ones. The varieties $\mathbb{A}^2$, $\mathbb{A}^2/G$ and $\mathscr{M}_C$ are toric (see for example [13, Chapter 10] or [25, Chapter 2]) and we can rewrite the diagram

$$\begin{tikzcd} & \mathbb{A}^2 \arrow[d, "\pi"] \\ \mathscr{M}_C \arrow[r, "\varepsilon_C"] & \mathbb{A}^2/G \end{tikzcd}$$

in terms of fans as follows:



In particular, $\mathscr{M}_C$ is covered by the k toric charts $U_j \cong \mathbb{A}^2$, for $j = 1, \ldots, k$, associated to the maximal cones of the chosen fan for $\mathscr{M}_C$ showed above.

Let us identify $\mathbb{A}^2/G$ with the subvariety of $\mathbb{A}^3$

$$\mathbb{A}^2/G = \left\{ (\alpha, \beta, \gamma) \in \mathbb{A}^3 \,\middle|\, \alpha\beta - \gamma^k = 0 \right\}$$

and let us put (toric) coordinates a_j, c_j on each U_j for $j = 1, \ldots, k$. Then, we can encode the diagram above into the following k diagrams

(1.1.3)
$$\begin{tikzcd} & \mathbb{A}^2 \arrow[d, "\pi"] & (x, y) \arrow[d, mapsto] \\ U_j \arrow[r, "\varepsilon_j"] & \mathbb{A}^2/G & (x^k, y^k, xy) \\ (a_j, c_j) \arrow[r, mapsto] & (a_j^{k-j+1} c_j^{k-j}, a_j^{j-1} c_j^{j}, a_j c_j) & \end{tikzcd}$$

for $j = 1,\dots,k$. In particular, we obtain some relations between the coordinates x, y on $\mathbb{A}^2$ and the coordinates a_j, c_j on U_j, namely

$$\begin{aligned} a_j &= x^j y^{j-k}, \\ c_j &= x^{1-j} y^{k-j+1}. \end{aligned} \tag{1.1.4}$$

Formally, these are relations between rational functions defined on $\mathbb{A}^2 \underset{\mathbb{A}^2/G}{\times} U_j$.

Remark 1.1.1. The toric points of each of the $k-1$ irreducible components of the exceptional divisor of the crepant resolution are origins of two consecutive charts, i.e. U_j and U_{j+1} for some $j = 1,\dots,k$. Therefore, we can order the collection of the components of the exceptional divisor by saying that the j-th component is the curve whose toric points are the origins of U_j and U_{j+1}.

1.1.3 The partial resolutions of the A_{k-1} singularities

Let $X = \left\{ xy - z^k = 0 \right\} \cong \mathbb{A}^2/(\mathbb{Z}/k\mathbb{Z})$ be a singularity of type A_{k-1}. It is an easy exercise in algebraic (or toric) geometry to show that the blowup $\overline{X}$ of X at the origin gives a partial resolution

$$Y \xrightarrow{\overline{f}} \overline{X} \xrightarrow{\overline{g}} X,$$

with the property that the strict transform of the exceptional locus of $\overline{g}$ via $\overline{f}$ consists of the first and the last exceptional curves (see Remark 1.1.1) of the resolution ε. In paricular, $\overline{X}$ has an isolated singularity of type A_{k-3}. Similarly one can show that the blowups with centre the ideals (x) and (y) give partial resolutions which pop up the first and last curve respectively and which have an isolated singularity of type A_{k-2}.

We show now how to compute all the other crepant resolutions. Consider the ideals $I_j \subset \mathbb{C}[X]$, for $j = 1,\dots,k-1$, defined by

$$I_j = (x, z^j) = (y, z^{k-j}) \quad \text{for all } j = 1,\dots,k-1,$$

where the equality follows from

$$X = \left\{ \det \begin{pmatrix} x & z^j \\ z^{k-j} & y \end{pmatrix} = 0 \right\} \quad \text{for all } j = 1,\dots,k-1.$$

Then, there is a partial resolution

$$Y \xrightarrow{f_j} X_j = \mathrm{Bl}_{I_j} X \xrightarrow{g_j} X.$$

More precisely, this is the partial resolution with the property that the strict transform via f_j of the exceptional locus of g_j is the j-th curve (see Remark 1.1.1) of the exceptional locus of $g_j \circ f_j$. This can be understood by looking at the equations of the blowup:

$$X_j = \left\{ u_1 y - z^{k-j} u_0 = x u_0 - z^j u_1 = 0 \right\} \subset \mathbb{A}^3 \times \mathbb{P}^1. \tag{1.1.5}$$

Indeed, form Equation (1.1.5) it is clear that X_j has an irreducible (rational) exceptional divisor over which lie the two singularities of X_j. These singularities are respectively of type A_{j-1} and A_{k-j-1}, which implies that X_j is the required partial resolution.

All the other crepant resolutions have of the form (see Lemma 0.2.1)

$$Y \xrightarrow{f_{i_1,\ldots,i_k}} X_j = \mathrm{Bl}_{\prod_{j=1}^{k} I_{i_j}} X \xrightarrow{g_{i_1,\ldots,i_k}} X,$$

for some $1 \le i_1 < \ldots < i_k \le k-1$. Each of them can be defined as the partial resolution with the property that the strict transform via $f_{i_1,\ldots,i_k}$ of the exceptional locus of $g_{i_1,\ldots,i_k}$ is $C_{i_1} \cup \cdots \cup C_{i_k}$ where we have denoted by C_i the i-the curve in the exceptional locus $\mathrm{Exc}(\varepsilon)$ of the crepant resolution of X.

1.2 Toric G-constellations

This section is devoted to the study of toric G-constellations, i.e. those G-constellations which, in addition to being G-sheaves, are also $\mathbb{T}^2$-sheaves. As it usually happens when dealing with $\mathbb{T}^2$-modules, we will see that the $\mathbb{C}[x, y]$-module structure of a toric G-constellation is fully described in terms of combinatorial objects, which in this case are called skew Ferrers diagrams.

This way of proceeding in the description of a $\mathbb{T}^2$-module is not new, and it is actually adopted very often in the literature; for example in the study of monomial ideals (see [8]) or, more generally, in the study of $\mathbb{T}^2$-modules of finite length (see [51]).

Although many statements can be generalized to higher dimension, from now on we will focus on the 2-dimensional case.

1.2.1 The torus action

Recall that $\mathbb{A}^2$ is a toric variety via the standard torus action:

$$\begin{aligned} \mathbb{T}^2 \times \mathbb{A}^2 &\longrightarrow \mathbb{A}^2 \\ ((\sigma_1,\sigma_2),(x,y)) &\longmapsto (\sigma_1\cdot x,\sigma_2\cdot y). \end{aligned}$$

Notice that, under our assumptions, G is a finite subgroup of the torus $\mathbb{T}^2$. Hence, the action of $\mathbb{T}^2$ commutes with the action of the finite abelian (diagonal) subgroup $G \subset \mathbb{T}^2$.

This implies that, given a θ-stable G-constellation $\mathscr{F}$, the pullback $\sigma^*\mathscr{F}$ is a θ-stable G-constellation. Indeed, σ^* induces an isomorphism between the global sections of $\sigma^*\mathscr{F}$ and $\mathscr{F}$ and hence, $\dim H^0(\mathbb{A}^2,\sigma^*\mathscr{F}) = k$. Moreover, $\sigma^*\mathscr{F}$ is still a G-sheaf if we define (see Chapter 0), $\forall g \in G$, the morphisms $\lambda_g^{\sigma^*\mathscr{F}} : \sigma^*\mathscr{F} \to g^*\sigma^*\mathscr{F}$ as

$$\lambda_g^{\sigma^*\mathscr{F}} = \sigma^*\lambda_g^{\mathscr{F}}.$$

Such morphisms are well defined because σ^* and g^* commute, i.e. $g^*\sigma^*\mathscr{F} \cong \sigma^*g^*\mathscr{F}$ for all $(g,\sigma) \in G \times \mathbb{T}^2$. Finally, we have to check that $\sigma^*\mathscr{F}$ is θ-stable. This follows from the fact that both the groups $G \subset \mathbb{T}^2$ act diagonally. As a consequence, if $\mathscr{E} \subset \mathscr{F}$ is a proper G-subsheaf and

$$H^0(\mathbb{A}^2,\mathscr{E}) = \bigoplus_{j=1}^{r} \rho_{i_j}$$

as representations (see Equation (1.1.2) for the definition of ρ_i), then $\sigma^*\mathscr{E} \subset \sigma^*\mathscr{F}$ is a proper G-subsheaf and

$$H^0(\mathbb{A}^2,\sigma^*\mathscr{E}) = \bigoplus_{j=1}^{r} \rho_{i_j}$$

as representations.

Definition 1.2.1. As explained above, the torus $\mathbb{T}^2$ acts on $\mathscr{M}_C$ for any chamber C. We will say that a G-constellation $\mathscr{F}$ is *toric* if it corresponds to a torus fixed point.

Remark 1.2.2. A G-constellation $\mathscr{F}$ is *toric* if and only if it is a $\mathbb{T}^2$-sheaf. Indeed, $\mathscr{F}$ is a torus fixed points if and only if, for all $\sigma \in \mathbb{T}^2$ there are isomorphisms $\psi_\sigma : \mathscr{F} \to \sigma^*\mathscr{F}$ and these isomorphisms provide the structure of $\mathbb{T}^2$-sheaf on $\mathscr{F}$ (see Chapter 0).

Definition 1.2.3. We will say that a G-constellation $\mathscr{F}$ is *nilpotent* if the endomorphisms $x\cdot$ and $y\cdot$ of the $\mathbb{C}[x,y]$-module $H^0(\mathbb{A}^2,\mathscr{F})$ are nilpotent.

Remark 1.2.4. A G-constellation $\mathscr{F}$ is supported on the origin $0\in\mathbb{A}^2$ if and only if it is nilpotent. This follows from the relation between the annihilator of a $\mathbb{C}[x,y]$-module and the support of the sheaf associated to it (see [20, Section 2.2]). Moreover, Theorem 1.0.21 implies that nilpotent C-stable G-constellations correspond to points of the exceptional locus of the crepant resolution $\mathscr{M}_C$.

Remark 1.2.5. Given a G-constellation $F = H^0(\mathbb{A}^2,\mathscr{F})$, we can compare its structures of representation and of $\mathbb{C}[x,y]$-module. Looking at the induced action of G on $\mathbb{C}[x,y]$, it turns out that, if $s\in\rho_i$ via the isomorphism $F\cong\mathbb{C}[G]$ then:

$$x\cdot s\in\rho_{i+1},$$

and,

$$y\cdot s\in\rho_{i-1}.$$

Proposition 1.2.6. *If $F = H^0(\mathbb{A}^2,\mathscr{F})$ is a nilpotent G-constellation then the endomorphism $xy\cdot$ is the zero endomorphism.*

Proof. The G-constellation F is a k-dimensional $\mathbb{C}$-vector space. Let us pick a basis

$$\{v_0,\dots,v_{k-1}\}$$

of F such that, for all $i=0,\dots,k-1$, $v_i\in\rho_i$ under the isomorphism $F\cong\mathbb{C}[G]$. As in Remark 1.2.5, for all $i=0,\dots,k-1$, we have:

$$x\cdot v_i\in\rho_{i+1},$$

and,

$$y\cdot v_i\in\rho_{i-1}$$

where the indices are thought modulo k. In other words,

$$x\cdot v_i\in\operatorname{Span}(v_{i+1}) \text{ and } y\cdot v_i\in\operatorname{Span}(v_{i-1}).$$

Therefore, we get:

$$xy\cdot v_i\in\operatorname{Span}(v_i),\quad \forall i=0,\dots,k-1$$

i.e.

$$xy\cdot v_i=\alpha_i v_i,\text{ with }\alpha_i\in\mathbb{C},\quad\forall i=0,\dots,k-1.$$

Now, the nilpotency hypothesis implies that $\alpha_i=0$ for all $i=0,\dots,k-1$. □

Remark 1.2.7. If a G-constellation $F = H^0(\mathbb{A}^2,\mathscr{F})$ is toric, then, it is also nilpotent. Indeed, following the same logic as in the proof of Proposition 1.2.6, we have

$$x^k\cdot v_i=\alpha_i v_i,\text{ with }\alpha_i\in\mathbb{C},\quad\forall i=0,\dots,k-1,$$

but torus equivariancy implies $\alpha_i=0$ for all $i=0,\dots,k-1$.

1.2.2 Skew Ferrers diagrams and G-stairs

The advantage of working with toric G-constellations is that their spaces of global sections can be described in terms of monomial ideals whose data are described by mean of combinatorial objects.

We can associate, to each element of the natural plane $\mathbb{N}^2$, two labels: namely a monomial and an irreducible representation. We achieve this by saying that *a polynomial* $p \in \mathbb{C}[x,y]$ *belongs to an irreducible representation* ρ_i if

$$\forall g \in G, \quad g \cdot p = \rho_i(g)p$$

i.e. p is an eigenfunction for the linear map $g\cdot$ with the complex number $\rho_i(g)$ as eigenvector. In particular, with the notations in Section 1.1.1, the monomial $x^i y^j$ belongs to the irreducible representation ρ_{i-j} of the abelian group G, where the index is tought modulo k. According to this association, we can define the *representation tableau* $\mathscr{T}_G$ as

$$\mathscr{T}_G = \left\{ (i,j,t) \in \mathbb{N}^2 \times \operatorname{Irr}(G) \,\middle|\, i-j \equiv t \pmod{k} \right\} \subset \mathbb{N}^2 \times \operatorname{Irr}(G).$$

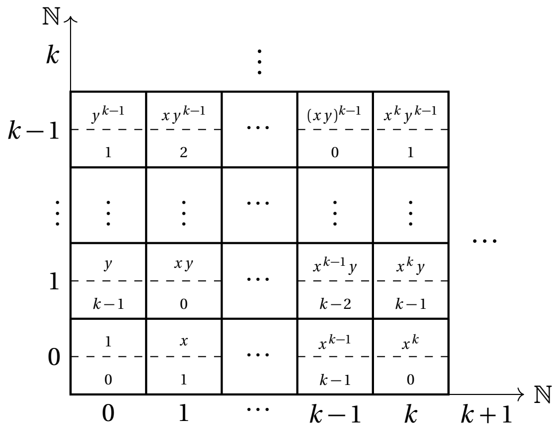


Figure 1.1. The representation tableau $\mathscr{T}_G$.

Notice that the labeling with the representation is superfluous because the first projection

$$\pi_{\mathbb{N}^2} : \mathscr{T}_G \longrightarrow \mathbb{N}^2$$

is a bijection. In any case, this notation is useful to keep in mind that we are dealing with the representation structure as well as with the module structure.

In summary, the representation tableau has the property that

(1.2.1) *moving to the right "increases" the irreducible representation by 1* $\pmod{k}$
moving up "decreases" the irreducible representation by 1 $\pmod{k}$.

Definition 1.2.8. A *Ferrers diagram (Fd)* is a subset A of the natural plane $\mathbb{N}^2$ such that

$$(\mathbb{N}^2 \smallsetminus A) + \mathbb{N}^2_+ \subset (\mathbb{N}^2 \smallsetminus A)$$

i.e. there exist $s \geq 0$ and $t_0 \geq \cdots \geq t_s \geq 0$ such that

$$A = \left\{ (i,j) \,\middle|\, i = 0,\ldots,s \text{ and } j = 0,\ldots,t_i \right\}.$$

Remark 1.2.9. In the literature there is some ambiguity about the name to be given to such diagrams. Indeed, sometimes, they are also called *Young tableaux* and, by Ferrers diagrams, something else is meant (for some different notations, see for example [27, 1]). In any case, we will adopt the notation in [16].

Pictorially, we see s consecutive columns of weakly decreasing heights. An example is depicted in Figure 1.2.

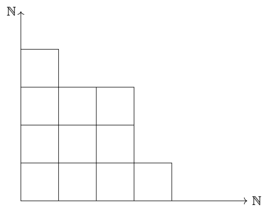


Figure 1.2. An example of Fd where $s=3$, $t_0=3$, $t_1=2$, $t_2=2$, $t_3=0$.

Remark 1.2.10. We briefly recall that, starting from a Ferrers diagram A, we can build a torus-invariant 0-dimensional subscheme Z of $\mathbb{A}^2$. Indeed, if $B=\mathbb{N}^2\smallsetminus A$ is the complement of A, then

$$I_Z=\left\{\, x^{b_1}y^{b_2}\,\middle|\,(b_1,b_2)\in B\,\right\}$$

is the ideal of the above mentioned subscheme $Z\subset\mathbb{A}^2$. In particular, the $\mathbb{C}[x,y]$-module structure of $H^0(\mathbb{A}^2,\mathscr{O}_Z)=\mathbb{C}[x,y]/I_Z$ is encoded in the Fd, by saying that a box, labeled by the monomial $m\in\mathbb{C}[x,y]$, corresponds to the 1-dimensional vector subspace of $H^0(\mathbb{A}^2,\mathscr{O}_Z)$ generated by m, and

$$\begin{array}{c}\textit{moving to the right in the Fd is the multiplication by } x\\ \textit{moving up in the Fd is the multiplication by } y.\end{array}\tag{1.2.2}$$

Definition 1.2.11. We will call *skew Ferrers diagram (sFd)* the set theoretic difference of two Ferrers diagrams.

Moreover, we will say that a sFd Γ is *connected* if, for any decomposition

$$\Gamma=\Gamma_1\cup\Gamma_2$$

as disjoint union, there are at least a box in Γ_1 and a box in Γ_2 which share an edge.

Lemma 1.2.12. *A skew Ferrers diagram Γ encodes the data of a torus-equivariant $\mathbb{C}[x,y]$-module M_Γ.*

Proof. Similarly as we did in Remark 1.2.10, we associate, to each Ferrers diagram A the ideal

$$I_A=\left(\left\{\, x^{b_1}y^{b_2}\in\mathbb{C}[x,y]\,\middle|\,(b_1,b_2)\in\mathbb{N}^2\smallsetminus A\,\right\}\right).$$

Suppose that $\Gamma=A_1\smallsetminus A_2$ is the difference of two Ferrers diagrams A_1,A_2. Then we can define the torus-equivariant $\mathbb{C}[x,y]$-module

$$M_\Gamma=I_{A_2}/I_{A_2}\cap I_{A_1}=I_{A_2}/I_{A_2\cup A_1}.$$

The fact that M_Γ does not depend on the decomposition $\Gamma = A_1 \setminus A_2$ follows noticing that, if we pick another decomposition $\Gamma = A'_1 \setminus A'_2$, then the isomorphism of $\mathbb{C}$-vector spaces

$$I_{A_2}/I_{A_2} \cap I_{A_1} \longrightarrow I_{A'_2}/I_{A'_2} \cap I_{A'_1},$$

which associates the class $x^\alpha y^\beta + I_{A_2} \cap I_{A_1}$ to the class $x^\alpha y^\beta + I_{A'_2} \cap I_{A'_1}$, is an isomorphism of $\mathbb{C}[x, y]$-modules. □

Now, instead of focusing just on subsets of the natural plane $\mathbb{N}^2$, we will introduce more structure by looking at subsets of the representation tableau.

In some instances, we will need to work with abstract sFd's obtained forgetting about the monomials.

Definition 1.2.13. We will call *G-sFd* a subset $A \subset \mathscr{T}_G$ of the representation tableau whose image $\pi_{\mathbb{N}^2}(A)$, under the first projection

$$\pi_{\mathbb{N}^2} : \mathscr{T}_G \longrightarrow \mathbb{N}^2,$$

is a sFd.

An *abstract G-sFd* is a diagram Γ made of boxes labeled by the irreducible representations of G that can be embedded into the representation tableau as a G-sFd.

Example 1.2.14. Consider the action $\mathbb{Z}/3\mathbb{Z} \curvearrowright \mathbb{A}^2$. In Figure 1.3 are shown an abstract G-sFd and two of its possible realisations as G-sFd.

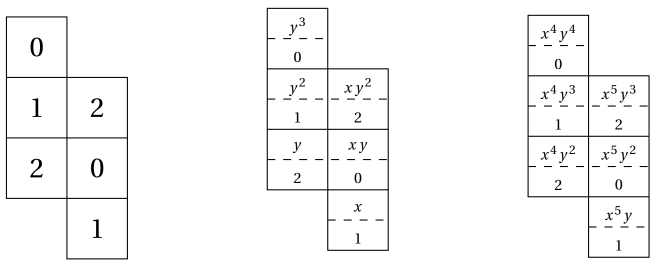


Figure 1.3. An abstract $\mathbb{Z}/3\mathbb{Z}$-sFd and two of its possible realisations as $\mathbb{Z}/3\mathbb{Z}$-sFd.

On the other hand, the diagram in Figure 1.4 is not an abstract G-sFd because it does not satisfy the rules (1.2.1) and (1.2.2).

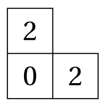


Figure 1.4.

Remark 1.2.15. Given any subset Ξ of the representation tableau and any monomial $x^\alpha y^\beta$ we will denote by $x^\alpha y^\beta \cdot \Xi$ the subset of the representation tableau obtained by translating Ξ α steps to the right and β steps up. Notice that this is compatible with the association $\mathbb{N}^2 \longleftrightarrow$ {monomials in two variables} as explained in Remark 1.2.10.

Lemma 1.2.16. *If $\mathscr{F}$ is a toric G-constellation then there exists a basis $\{v_0,\dots,v_{k-1}\}$ of $F = H^0(\mathbb{A}^2,\mathscr{F})$ such that*

1. *for all $i = 0,\dots,k-1$, $v_i \in \rho_i$,*
2. *for all $i = 0,\dots,k-1$, v_i are semi-invariant functions with respect some character χ_i of $\mathbb{T}^2$, i.e. $(a,b)\cdot v_i = \chi_i(a,b)v_i$ for all $(a,b)\in\mathbb{T}^2$,*
3. *for all $i = 0,\dots,k-1$,*

$$\begin{cases} x\cdot v_i \in \{v_{i+1},0\},\\ y\cdot v_i \in \{v_{i-1},0\}.\end{cases}$$

Proof. We can always pick a basis $\{\widetilde{v}_0,\dots,\widetilde{v}_{k-1}\}$ which satisfies *(1)* and *(2)*. Moreover, it follows from Remark 1.2.5 that:

$$\begin{cases} x\cdot \widetilde{v}_i \in \mathrm{Span}(\widetilde{v}_{i+1}),\\ y\cdot \widetilde{v}_i \in \mathrm{Span}(\widetilde{v}_{i-1}),\end{cases}$$

where the indices are thought modulo k. The fact that $\mathscr{F}$ is toric implies that there are no “cycles", i.e. there are no $1<s<k$ and

$$\left\{ (i_j,k_j,h_j,\sigma_j)\in \mathrm{Irr}(G)\times\mathbb{N}^2\times\mathbb{C}^* \;\middle|\; \begin{array}{c} j=1,\dots,s,\\ i_j\neq i_{j'} \text{ for } j\neq j',\\ k_j+h_{j+1}>0\end{array}\right\}$$

where the indices are thought modulo s, such that

$$\begin{cases} (x\cdot)^{k_1}\widetilde{v}_{i_1} &= \sigma_1(y\cdot)^{h_2}\widetilde{v}_{i_2},\\ (x\cdot)^{k_2}\widetilde{v}_{i_2} &= \sigma_2(y\cdot)^{h_3}\widetilde{v}_{i_3},\\ &\vdots\\ (x\cdot)^{k_{s-1}}\widetilde{v}_{i_{s-1}} &= \sigma_{s-1}(y\cdot)^{h_s}\widetilde{v}_{i_s},\\ (x\cdot)^{k_s}\widetilde{v}_{i_s} &= \sigma_s(y\cdot)^{h_1}\widetilde{v}_{i_1}.\end{cases} \tag{1.2.3}$$

Indeed, x and y are semi-invariant functions with respect to the characters

$$\begin{aligned} \mathbb{T}^2 &\xrightarrow{\lambda_x} \mathbb{C}^*\\ (a,b) &\longmapsto a\end{aligned}$$

and

$$\begin{aligned} \mathbb{T}^2 &\xrightarrow{\lambda_y} \mathbb{C}^*\\ (a,b) &\longmapsto b\end{aligned}$$

of the torus $\mathbb{T}^2$. Then, if we act on both sides of the Equations 1.2.3 with some $(a,b)\in\mathbb{T}^2$, we get:

$$
\begin{cases}
\lambda_x(a,b)^{k_1}\chi_{i_1}(a,b)(x\cdot)^{k_1}\widetilde{v}_{i_1}=\sigma_1\lambda_y(a,b)^{h_2}\chi_{i_2}(a,b)(y\cdot)^{h_2}\widetilde{v}_{i_2},\\
\lambda_x(a,b)^{k_2}\chi_{i_2}(a,b)(x\cdot)^{k_2}\widetilde{v}_{i_2}=\sigma_2\lambda_y(a,b)^{h_3}\chi_{i_3}(a,b)(y\cdot)^{h_3}\widetilde{v}_{i_3},\\
\vdots\\
\lambda_x(a,b)^{k_{s-1}}\chi_{i_{s-1}}(a,b)(x\cdot)^{k_{s-1}}\widetilde{v}_{i_{s-1}}=\sigma_{s-1}\lambda_y(a,b)^{h_s}\chi_{i_s}(a,b)(y\cdot)^{h_s}\widetilde{v}_{i_s},\\
\lambda_x(a,b)^{k_s}\chi_{i_s}(a,b)(x\cdot)^{k_s}\widetilde{v}_{i_s}=\sigma_s\lambda_y(a,b)^{h_1}\chi_{i_1}(a,b)(y\cdot)^{h_1}\widetilde{v}_{i_1},
\end{cases}
\tag{1.2.4}
$$

Now, the System 1.2.4 is equivalent to:

$$
\begin{cases}
a^{k_1}\chi_{i_1}(a,b)=b^{h_2}\chi_{i_2}(a,b),\\
a^{k_2}\chi_{i_2}(a,b)=b^{h_3}\chi_{i_3}(a,b),\\
\vdots\\
a^{k_{s-1}}\chi_{i_{s-1}}(a,b)=b^{h_s}\chi_{i_s}(a,b),\\
a^{k_s}\chi_{i_s}(a,b)=b^{h_1}\chi_{i_1}(a,b),
\end{cases}
$$

which is equivalent to

$$
a^{k_1+\cdots+k_s}=b^{h_1+\cdots+h_s}\quad\forall(a,b)\in\mathbb{T}^2.
\tag{1.2.5}
$$

Finally, the only solution of Equation (1.2.5) is

$$
k_1=\cdots=k_s=h_1=\cdots=h_s=0,
$$

which contradicts the hypothesis $k_i+h_{i+1}>0$ for all $i=1,\dots,s$.

We are now ready to build the requested basis. Let $\{w_1,\dots,w_\ell\}\subset\{\widetilde{v}_0,\dots,\widetilde{v}_{k-1}\}$ be a minimal set of generators of the $\mathbb{C}[x,y]$-module F, i.e. the set

$$
\left\{\, w_j+\mathfrak{m}\cdot F\in F/\mathfrak{m}\cdot F \,\middle|\, j=1,\dots,\ell \,\right\}
$$

is a basis of the $\mathbb{C}$-vector space $F/\mathfrak{m}\cdot F$. Let us also denote by F_j, for $j=1,\dots,\ell$, the submodule generated by w_j. We start by taking, for all $j=1,\dots,\ell$, as basis of F_j the set

$$
B_j=\left\{\, x^\alpha y^\beta w_j \,\middle|\, \alpha\cdot\beta=0 \,\right\}.
$$

The problem is that in general the union of all B_j's is not a basis of F because there can be some relations $x^\alpha w_i=\mu y^\beta w_j$ for $i\neq j$ and $\mu\in\mathbb{C}^*\smallsetminus 1$. The fact that there are no cycles implies that we can re-scale all the elements in each B_j obtaining new $\overline{B}_j$ so that $\bigcup_j \overline{B}_j$ is a basis of F that verifies properties *(1)*, *(2)*, *(3)*. □

Proposition 1.2.17. *Given a, possibly reducible, toric G-constellation $F=H^0(\mathbb{A}^2,\mathscr{F})$, there is (at least) one G-sFd whose associated $\mathbb{C}[x,y]$-module is a G-constellation isomorphic to F.*

Remark 1.2.18. If we find one G-sFd with the required property, then there are infinitely many of them. Indeed, a special property of the representation tableau is that translations enjoy some periodicity properties.

Let Γ be a G-sFd, then:

1. multiplication by x has period k, i.e there is an isomorphism of $\mathbb{C}[x,y]$-modules
$$M_\Gamma \xrightarrow{\sim} M_{x^k\cdot\Gamma}$$
which induces an isomorphism of representations between M_Γ and $M_{x^k\cdot\Gamma}$;

2. multiplication by y has period k, i.e there is an isomorphism of $\mathbb{C}[x,y]$-modules
$$M_\Gamma \xrightarrow{\sim} M_{y^k\cdot\Gamma}$$
which induces an isomorphism of representations between M_Γ and $M_{y^k\cdot\Gamma}$;

3. multiplication by xy is an isomorphism, i.e there is an isomorphism of $\mathbb{C}[x,y]$-modules
$$M_\Gamma \xrightarrow{\sim} M_{xy\cdot\Gamma}$$
which induces an isomorphism of representations between M_Γ and $M_{xy\cdot\Gamma}$.

In particular, all these G-sFd's correspond to the same abstract G-sFd.

Proof. (*of Proposition 1.2.17*). Let $\{v_0,\dots,v_{k-1}\}$ be a $\mathbb{C}$-basis of F with the properties listed in Lemma 1.2.16, and let $\{\, w_j = v_{i_j} \mid j = 1,\dots,s\,\}$ be a minimal set of generators of F as a $\mathbb{C}[x,y]$-module (see the proof of Lemma 1.2.16). Denote by F_j, for $j=1,\dots,s$, the $\mathbb{C}[x,y]$-submodule of F generated by w_j. We can represent each F_j by using diagrams of the form shown in Figure 1.5,

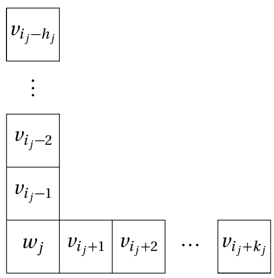


Figure 1.5.

where the integers k_j and h_j are defined by
$$k_j = \max\left\{\, \alpha \,\middle|\, (x\cdot)^\alpha w_j \neq 0 \,\right\}$$
and
$$h_j = \max\left\{\, \alpha \,\middle|\, (y\cdot)^\alpha w_j \neq 0 \,\right\},$$
and they are well defined because any toric G-constellation is nilpotent by Remark 1.2.7.

The $\mathbb{C}[x,y]$-module structure of F_j is encoded in the fact that the multiplication by x (resp. y) sends the generator of a box (i.e., the generator of the corresponding vector space) to the generator of the box on the left (resp. above). If there is no box on the left (resp. above) this means that the multiplication by x (resp. y) is zero.

Now, we have to glue these diagrams to form the required G-sFd. We will glue them along boxes with the same labels. First, notice that, if, for some $j \neq j'$ and $r, t \geq 1$, we have $(x\cdot)^r w_j = (x\cdot)^t w_{j'}$, i.e. $i_j + r = i_{j'} + t$ modulo k, then

$$(x\cdot)^r w_j = (x\cdot)^t w_{j'} = 0.$$

Indeed, if $r < t$ (the case $r \geq t$ is analogous) then, a representation argument (see Proposition 1.2.6) tells us that $w_j = (x\cdot)^{t-r} w_{j'}$ which, whenever $(x\cdot)^r w_i \neq 0$, contradicts the minimality of the generating set $\{w_1, \dots, w_s\}$. Analogously, if, for some $j \neq j'$ and $r, t \geq 1$, we have $(y\cdot)^r w_j = (y\cdot)^t w_{j'}$, then $(y\cdot)^r w_j = 0$.

Now we show that, if, for some $j \neq j'$ and $r, t \geq 1$, we have $(x\cdot)^r w_j = (y\cdot)^t w_{j'}$, then $r = k_j$ and $t = h_{j'}$. Suppose, by contradiction, that there exists $1 \leq r < k_j$ such that $(x\cdot)^r w_j = (y\cdot)^t w_{j'}$ (the case $1 \leq t < h_{j'}$ is similar). In particular, the minimality assumption implies $t \geq 1$. Since $r < k_j$, by definition of k_j, we have $(x\cdot)^{r+1} w_i \neq 0$. Therefore, we get

$$0 \neq (x\cdot)^{r+1} w_j = x \cdot ((x\cdot)^r w_j) = x \cdot y^t \cdot w_{j'} = (xy) \cdot y^{t-1} \cdot w_{j'} = 0$$

which gives a contradiction.

The last thing to be checked is that there are no “cycles". Explicitly, suppose that, up to reordering the $v_i's$, and consequently the $w_i's$, we already glued ℓ diagrams, as above, of the form depicted in Fig. 1.5 to a diagram of the form shown in Figure 1.6,

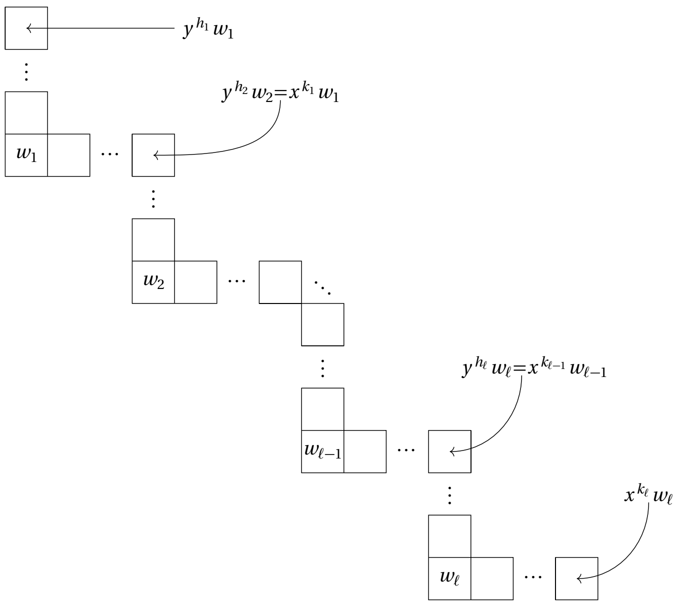


Figure 1.6.

we want to show that there is no gluing $(x\cdot)^{k_\ell} w_\ell = \sigma (y\cdot)^{h_1} w_1$ for some $\sigma \in \mathbb{C}^*$, i.e. no gluing of the first and the last boxes of the above diagram. The presence of this cycle would translate into the following system of equalities

$$\begin{cases} (x\cdot)^{k_1} w_1 & = (y\cdot)^{h_2} w_2, \\ (x\cdot)^{k_2} w_2 & = (y\cdot)^{h_3} w_3, \\ & \vdots \\ (x\cdot)^{k_{\ell-1}} w_{\ell-1} & = (y\cdot)^{h_\ell} w_\ell, \\ (x\cdot)^{k_\ell} w_\ell & = \sigma(y\cdot)^{h_1} w_1, \end{cases}$$

which cannot be verified by any toric G-constellation as explained in the proof of Lemma 1.2.16.

So far we have proven that each connected component of the required G-sFd has the shape depicted in Figure 1.7.

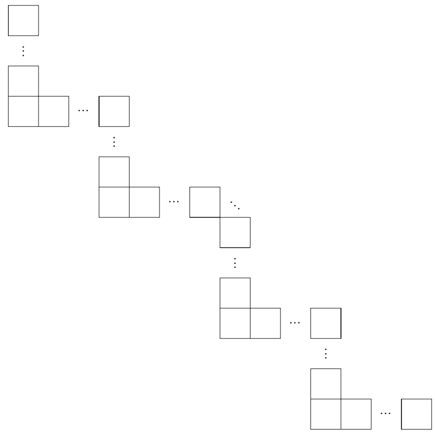


Figure 1.7.

Moreover, if we forget about the reordering, each box will contain a label v_i whose index increases by one when moving to the right or downward in the diagram. Since we have chosen $v_i \in \rho_i$ for $i = 0, \dots, k-1$, this diagram fits in the representation tableau (see Section 1.2.2), i.e. it is an abstract G-sFd. After performing all possible gluings, we obtain a number of abstract G-sFd's $A_1, \dots, A_m$ whose shape is drawn in Figure 1.7.

The last thing to do is to show that we can realize $A_1, \dots, A_m$ as subsets $\Gamma_1, \dots, \Gamma_m$ of the representation tableau to get a G-sFd, i.e. in such a way that

$$\pi_{\mathbb{N}^2}\left(\bigcup_{i=1}^{m} \Gamma_i\right)$$

is a sFd. This can be done in many ways and we explain one possible way to proceed.

We start by realizing $A_1, \dots, A_m$ as disjoint G-sFd's $\Gamma_1, \dots, \Gamma_m$. This can always be done because, as we observed, $A_1, \dots, A_m$ are abstract G-sFd's and, from any choice of realisations $\widetilde{\Gamma}_1, \dots, \widetilde{\Gamma}_m$ of them as non-necessarily disjoint G-sFd's, we can obtain disjoint $\Gamma_1, \dots, \Gamma_m$ by performing the translations described in Remark 1.2.18.

At this point, we have m disjoint G-sFd's as described in Figure 1.8,

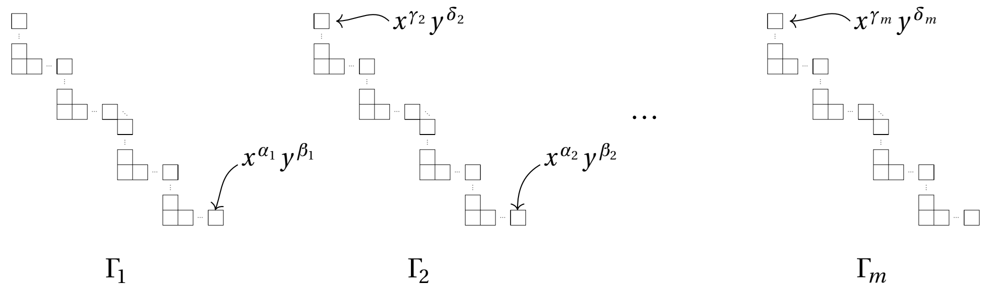


Figure 1.8.

where just the labels of the boxes we are interested in are shown. The problem is that, in general, the union $\bigcup_{i=1}^{m}\Gamma_i$ is not a G-sFd, i.e. $\pi_{\mathbb{N}^2}\left(\bigcup_{i=1}^{m}\Gamma_i\right)$ is not a sFd. In order to solve this problem, we have to perform some translations. The required G-sFd is

$$\Gamma = \bigcup_{i=1}^{m}\overline{\Gamma}_i,$$

where

$$\overline{\Gamma}_i = x^{k\sum_{j=1}^{i-1}\alpha_j}\, y^{k\sum_{j=1+i}^{m}\delta_j}\cdot\Gamma_i \quad \text{for } i=1,\dots,m.$$

The proof that Γ is a G-sFd is now an easy check. $\square$

As a byproduct of the proof, we also get that any G-sFd associated to an irreducible toric G-constellation has a particular shape.

Definition 1.2.19. We will say that a a connected G-sFd Γ is a *stair* if

$$(m,n)\in\pi_{\mathbb{N}^2}(\Gamma)\Rightarrow(m+1,n+1),(m-1,n-1)\notin\pi_{\mathbb{N}^2}(\Gamma).$$

Moreover,

- we will call G*-stair* a stair made of k boxes,
- we will call *abstract (G-)stair* an abstract G-sFd whose realisation in the representation tableau is a (G-)stair,
- given a stair Γ we will call *(anti)generators* of Γ the boxes positioned in the (top) lower corners of Γ (see Figure 1.9),
- we will call substair any (even not connected) subset of a stair.

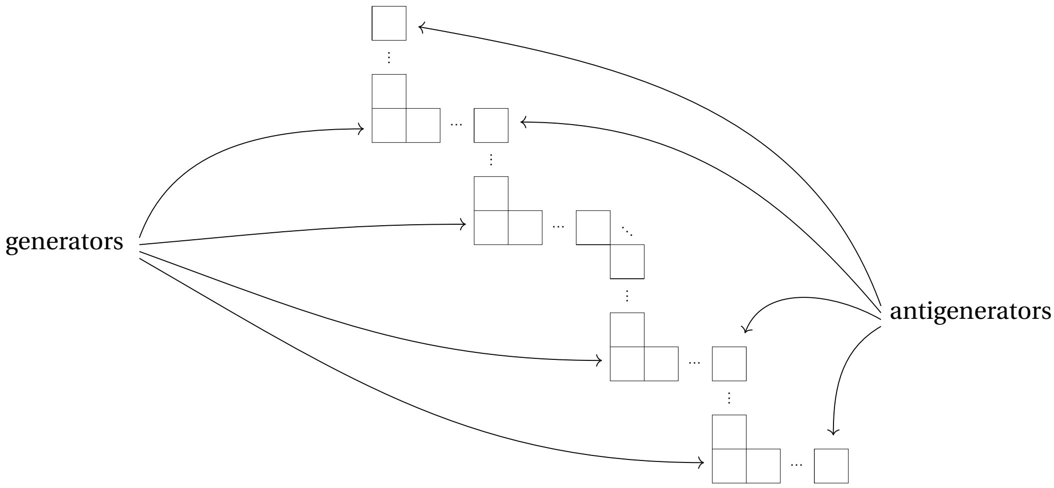


Figure 1.9. Generators and antigenerators of a stair.

Remark 1.2.20. If $\mathscr{F}$ is any toric G-constellation, and $\Gamma_{\mathscr{F}}$ is any G-sFd associated to $\mathscr{F}$, then $\Gamma_{\mathscr{F}}$ is connected, i.e. it is a G-stair, if and only if $\mathscr{F}$ is irreducible.

In this case we will refer to the upper left box as the *first box* and we will refer to the lower right box as the *last box*. In this a way, we provide of an order the boxes of a G-stair and, consequently, we provide of an order also the irreducible representations of G.

Remark 1.2.21. The set of generators of a stair Γ corresponds to a minimal set of generators of the $\mathbb{C}[x,y]$-module M_Γ associated to Γ, i.e. $m_1,\dots,m_s \in M_\Gamma$ such that

$$\{\, m_i + \mathfrak{m}\cdot M_\Gamma \in M_\Gamma/\mathfrak{m}\cdot M_\Gamma \mid i=1,\dots,s \,\}$$

is a $\mathbb{C}$-basis of $M_\Gamma/\mathfrak{m}\cdot M_\Gamma$. Antigenerators correspond to 1-dimensional $\mathbb{C}[x,y]$-submodules of M_Γ, i.e. they form a $\mathbb{C}$-basis of the so-called socle

$$(0 :_{M_\Gamma} \mathfrak{m}) = \{\, m \in M_\Gamma \mid \mathfrak{m}\cdot m = 0 \in M_\Gamma \,\}.$$

Since each irreducible representation of G appears once in a G-stair L, sometimes, with abuse of notation, we will say that an irreducible representation is a (anti)generator for L.

Definition 1.2.22. Given a connected G-sFd Γ, we will call, respectively, *height* and *width* of Γ the integers $\mathfrak{h}(\Gamma)$ and $\mathfrak{w}(\Gamma)$ given by the height and the width of the smallest rectangle in $\mathbb{N}^2$ containing $\pi_{\mathbb{N}^2}(\Gamma)$.

Moreover, given an irreducible toric G-constellation $\mathscr{F}$, we will call, respectively, *height* and *width* of $\mathscr{F}$ the integers $\mathfrak{h}(\mathscr{F})$ and $\mathfrak{w}(\mathscr{F})$ given by the height and the width of any G-stair which represents $\mathscr{F}$.

We will see in Lemma 1.3.3 that to have a certain height (or width) prescribes the position of a G-constellation in the moduli spaces.

1.3 The chamber decomposition of Θ and the moduli spaces $\mathcal{M}_C$

This section is devoted to the proof of the first main result (Theorem 1.3.17) of the first chapter. In the first part of the section we will analyze the toric points of $\mathcal{M}_C$ and the corresponding G-constellations. Then, we will show how to construct 1-dimensional families of nilpotent G-constellations. Finally, in the last part, we will give the proof of the first main result.

1.3.1 The toric points of $\mathcal{M}_C$

Remark 1.3.1. The toric points of $\mathcal{M}_C$ are the origins of the charts U_j (see Equation (1.1.3)) and they correspond to the toric C-stable G-constellations. Indeed, the toric action that makes $\mathcal{M}_C$ a toric variety, as described in beginning of this section, coincides with the action

$$\begin{aligned} \mathcal{M}_C \times \mathbb{T}^2 &\longrightarrow \mathcal{M}_C \\ ([\mathcal{F}], \sigma) &\longmapsto [\sigma^* \mathcal{F}]. \end{aligned}$$

This is a consequence of the universal property of $\mathcal{M}_C$. Notice that, outside the exceptional locus of $\mathcal{M}_C$, i.e. on the open subset of free orbits, a direct computation is enough to show that the two actions agree.

As a consequence, we have a total order on the toric G-constellations over $\mathcal{M}_C$, in the sense that the first toric G-constellation is the G-constellation over the origin of U_1, the second one is the G-constellation over the origin of U_2, and so on.

Remark 1.3.2. Let Γ be a G-stair, then there exists a unique $\sigma \in \mathrm{Irr}(G)$ such that

$$y \cdot \sigma = 0 \text{ and } x \cdot \sigma \otimes \rho_{-1} = 0$$

in Γ. In particular, the representation σ corresponds to the first box of Γ. This representation is important because, if we want to deform in a non-trivial way the G-constellation $\mathcal{F}_\Gamma$ associated to Γ keeping the property of being nilpotent, there are only two ways to do it, namely to modify the $\mathbb{C}[x, y]$-module structure of $\mathcal{F}_\Gamma$ by imposing

$$y \cdot \sigma = \lambda \cdot \sigma \otimes \rho_{-1}, \quad \lambda \in \mathbb{C}^*$$

or

$$x \cdot \sigma \otimes \rho_{-1} = \mu \cdot \sigma, \quad \mu \in \mathbb{C}^*.$$

Indeed, if $y \cdot \sigma = \lambda \cdot \sigma \otimes \rho_{-1}$ is not zero, then the nilpotency hypothesis implies

$$x \cdot \sigma \otimes \rho_{-1} = \frac{1}{\lambda} x y \cdot \sigma = 0,$$

and the other case is similar. Comparing this with the proof of Lemma 1.2.16 one can show that letting λ (resp. μ) varying in $\mathbb{C}^*$ all the G-constellations so obtained are not isomorphic to each other (as G-constellations). In particular λ, μ are coordinates on a chart of $\mathcal{M}_C$ around $\mathcal{F}_\Gamma$.

As a consequence of the above remark, we obtain the following lemma.

Lemma 1.3.3. *If $\mathscr{F}_j$ is the toric G-constellation over the origin of the j-th chart of some $\mathscr{M}_C$, then we have*

$$\mathfrak{h}(\mathscr{F}_j) = k - j + 1$$

or, equivalently

$$\mathfrak{w}(\mathscr{F}_j) = j.$$

Proof. Let $\Gamma_j \subset \mathscr{T}_G$ be a G-stair for $\mathscr{F}_j$. In particular, it has the form in Figure 1.10 where just

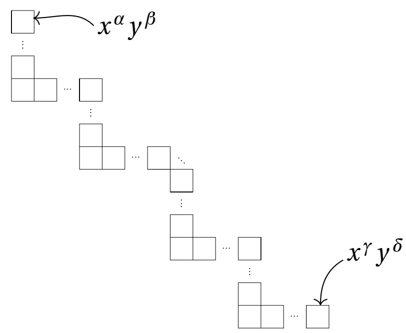


Figure 1.10.

the labels of the boxes we are interested in are shown. Recall, from Section 1.2.2, that, if we write the skew Ferrers diagram $\pi_{\mathbb{N}^2}(\Gamma_j) = A \smallsetminus B$ as the difference of two Ferrers diagrams A and B, then $\mathscr{F}_j \cong M_{\Gamma_j}$, where

$$M_{\Gamma_j} \cong \frac{I_A}{I_A \cap I_B},$$

and I_A, I_B are as in the proof of Lemma 1.2.12. Now, if we deform $\mathscr{F}_j$ as in Remark 1.3.2, by using the parameters $a_j, c_j \in \mathbb{C}$, we get relations:

$$x \cdot x^\gamma y^\delta = a_j x^\alpha y^\beta$$

$$y \cdot x^\alpha y^\beta = c_j x^\gamma y^\delta$$

and, the relations 1.1.4 tell us that

$$(\gamma - \alpha + 1, \delta - \beta) = (\mathfrak{w}(\mathscr{F}), -\mathfrak{h}(\mathscr{F}) + 1) = (j, j - k) \in \mathbb{N}^2$$

$$(\alpha - \gamma, \beta - \delta + 1) = (-\mathfrak{w}(\mathscr{F}) + 1, \mathfrak{h}(\mathscr{F})) = (1 - j, k - j + 1) \in \mathbb{N}^2$$

which completes the proof. $\square$

The previous lemma implies the following result.

Corollary 1.3.4. *Different toric G-constellations of the same height (or equivalently width) cannot belong to the same chamber.*

1.3.2 One-dimensional families

Definition 1.3.5. Given a G-constellation $\mathscr{F}$ and its abstract G-stair $\Gamma_{\mathscr{F}}$, we will call its *favorite condition* the stability condition $\theta_{\mathscr{F}} \in \Theta$ defined by:

$$(\theta_{\mathscr{F}})_i = \begin{cases} -2 & \text{if } \rho_i \text{ is a generator and it is neither the first nor the last box of } \Gamma_{\mathscr{F}}, \\ -1 & \text{if } \rho_i \text{ is a generator and it is either the first or the last box of } \Gamma_{\mathscr{F}}, \\ 2 & \text{if } \rho_i \text{ is an antigenerator and it is neither the first nor the last box of } \Gamma_{\mathscr{F}}, \\ 1 & \text{if } \rho_i \text{ is an antigenerator and it is either the first or the last box of } \Gamma_{\mathscr{F}}, \\ 0 & \text{otherwise} \end{cases}$$

Moreover, we will call the *cone of good conditions for* $\mathscr{F}$, the cone:

$$\Theta_{\mathscr{F}} = \left\{ \theta \in \Theta^{\text{gen}} \,\middle|\, \mathscr{F} \text{ is } \theta\text{-stable} \right\}.$$

Definition 1.3.6. Let Γ be a stair and let $\Gamma' \subset \Gamma$ be a substair. We will say that an element $v \in \Gamma'$ is

- a *left internal endpoint* of Γ' if there exists $w \in \Gamma \smallsetminus \Gamma'$ such that $x \cdot w = v$ or if $y \cdot v \in \Gamma \smallsetminus \Gamma'$;
- a *right internal endpoint* of Γ' if there exists $w \in \Gamma \smallsetminus \Gamma'$ such that $y \cdot w = v$ or if $x \cdot v \in \Gamma \smallsetminus \Gamma'$.

Moreover, we will say that

- a left (resp. right) internal endpoint is a *horizontal left (resp. right) cut* if $y \cdot v \in \Gamma \smallsetminus \Gamma'$ (resp. there exists $w \in \Gamma \smallsetminus \Gamma'$ such that $y \cdot w = v$);
- a left (resp. right) internal endpoint is a *vertical left (resp. right) cut* if there exists $w \in \Gamma \smallsetminus \Gamma'$ such that $x \cdot w = v$ (resp. $x \cdot v \in \Gamma \smallsetminus \Gamma'$);

Example 1.3.7. In Figure 1.11, the substair Γ has two internal endpoints, respectively a horizontal left cut and a vertical right cut, while Γ' has only one internal endpoint which is a vertical left cut.

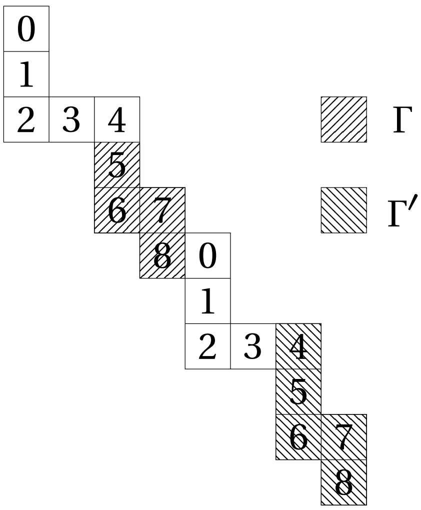


Figure 1.11.

Remark 1.3.8. If $\mathscr{F}$ is a G-constellation and $\Gamma_{\mathscr{F}}$ is a G-stair for $\mathscr{F}$, then a substair $\Gamma \subset \Gamma_{\mathscr{F}}$ corresponds to a G-equivariant $\mathbb{C}[x,y]$-submodule $\mathscr{E}_\Gamma$ of $\mathscr{F}$ if and only if it has only vertical left cuts and horizontal right cuts. Moreover, if Γ is connected and $\theta_{\mathscr{F}}$ is the favorite condition of $\mathscr{F}$, then,

$$\theta_{\mathscr{F}}(\mathscr{E}_\Gamma) = \begin{cases} 1 & \text{if } \Gamma \text{ has one internal endpoint,} \\ 2 & \text{if } \Gamma \text{ has two internal endpoints.} \end{cases}$$

Remark 1.3.9. Let $\mathscr{F}$ be a toric G-constellation with abstract G-stair $\Gamma_{\mathscr{F}}$ and let $\mathscr{E} < \mathscr{F}$ be a subrepresentation whose substair $\Gamma_{\mathscr{E}} \subset \Gamma_{\mathscr{F}}$ is connected. Then, if $\Gamma_{\mathscr{E}}$ has two horizontal cuts or two vertical cuts and $\theta_{\mathscr{F}}$ is the favorite condition of $\mathscr{F}$, we have

$$\theta_{\mathscr{F}}(\mathscr{E}) = 0.$$

Remark 1.3.10. The following properties are easy to check for a toric G-constellation $\mathscr{F}$:

- favorite conditions are never generic,
- the G-constellation $\mathscr{F}$ is $\theta_{\mathscr{F}}$-stable,
- there exist generic conditions $\theta \in \Theta^{\text{gen}}$ such that $\mathscr{F}$ is θ-stable, i.e. the cone of good conditions $\Theta_{\mathscr{F}}$ is not empty.

Moreover, given a chamber C, we have:

$$C = \bigcap_{[\mathscr{F}] \in \mathscr{M}_C} \Theta_{\mathscr{F}}.$$

We prove only the third property as, in what follows, we shall need similar arguments.

Let ρ_i be any irreducible representation, we will denote by $\mathscr{F}_{\rho_i}$ the G-equivariant $\mathbb{C}[x,y]$-submodule of $\mathscr{F}$ generated by ρ_i and, we will denote by $\Gamma_{\rho_i} \subset \Gamma_{\mathscr{F}}$ the abstract substair and G-stair corresponding to $\mathscr{F}_{\rho_i}$ and $\mathscr{F}$ respectively.

Consider an $\varepsilon \in \Theta$ with the following properties:

$$\begin{cases} \varepsilon_i = 0 & \text{if } \rho_i \text{ is an antigenerator,} \\ \varepsilon_i < 0 & \text{if } \rho_i \text{ is neither a generator nor an antigenerator,} \\ \varepsilon_i = -\sum\limits_{\rho_j \in (\Gamma_{\rho_i} \smallsetminus \rho_i)} \varepsilon_j & \text{if } \rho_i \text{ is a generator,} \\ \sum\limits_{\rho_i \text{ generator}} \varepsilon_i < 1. \end{cases}$$

Then, for any subrepresentation $\mathscr{E} < \mathscr{F}$, we have

$$\varepsilon(\mathscr{E}) > -\sum_{\rho_i \text{ generator}} \varepsilon_i > -1.$$

Hence, the G-constellation $\mathscr{F}$ is $(\theta_{\mathscr{F}} + \varepsilon)$-stable. Indeed, Remark 1.3.8 implies that, given an irreducible proper G-equivariant $\mathbb{C}[x,y]$-submodule we have

$$(\theta_{\mathscr{F}} + \varepsilon)(\mathscr{E}) > 0.$$

On the contrary, if $\mathscr{E}$ is not irreducible then it is a direct sum of irreducible components and $(\theta_{\mathscr{F}}+\varepsilon)(\mathscr{E})>0$ follows by the additivity of $\theta_{\mathscr{F}}+\varepsilon$ on direct sums.

We conclude by noticing that $\Theta\smallsetminus\Theta^{\mathrm{gen}}$ is a union of hyperplanes and so, there is at least a choice $\varepsilon\in\Theta$ such that $\theta_{\mathscr{F}}+\varepsilon$ is generic.

We will see in the proof of Theorem 1.3.17 that there is an easier way, which does not involve any ε, to prove that $\Theta_{\mathscr{F}}$ is not empty.

Definition 1.3.11. An *abstract linking stair* is an abstract stair made of $2k$ boxes obtained from an abstract G-stair Γ in either of the following ways:

1. (*decreasing* linking stair of Γ) take two copies of Γ and make a new abstract stair by gluing the right edge of the last box of one copy to the left edge of the first box of the other copy;

2. (*increasing* linking stair of Γ) take two copies of Γ and make a new abstract stair by gluing the lower edge of the last box of one copy to the upper edge of the first box of the other copy.

A *linking stair* is a realisation of an abstract linking stair as a subset of the representation tableau.

Remark 1.3.12. An abstract linking stair contains exactly k different abstract G-stairs.

Proposition 1.3.13. *Let Γ be the abstract G-stair of a G-constellation $\mathscr{F}$ and let L be its abstract decreasing linking stair. Consider any G-stair $\Gamma'\subset L$ and its associated G-constellation $\mathscr{F}'$. Then, the following are equivalent:*

1. *there exists at least a chamber C such that both $\mathscr{F}$ and $\mathscr{F}'$ belong to C, i.e. $\Theta_{\mathscr{F}}\cap\Theta_{\mathscr{F}'}\neq\emptyset$,*

2. *$\mathfrak{h}(\mathscr{F}')=\mathfrak{h}(\mathscr{F})-1$,*

3. *the substair $\Gamma'\subset L$ has a horizontal left cut.*

In particular, $\mathscr{F}'$ is the G-constellation next to $\mathscr{F}$ in $\mathscr{M}_C$ as per Remark 1.3.1.

Example 1.3.14. Figure 1.12 describes the situation via an example. Here, we are considering the action $\mathbb{Z}/9\mathbb{Z}\curvearrowright\mathbb{A}^2$ (see Equation (1.1.1)).

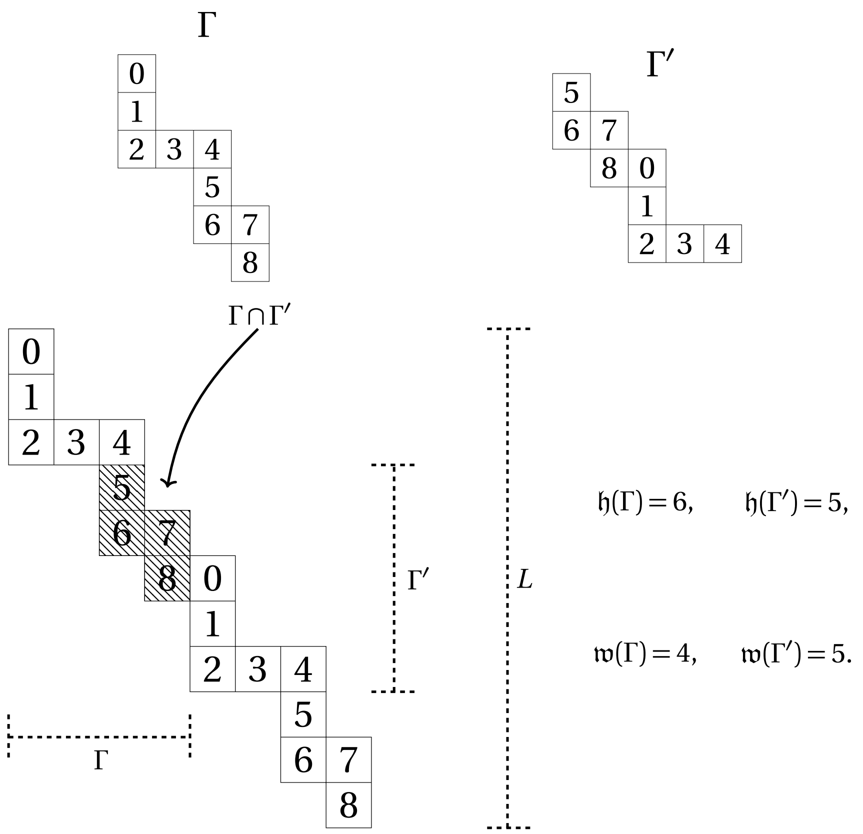


Figure 1.12. The abstract linking stair L of an abstract G-stair Γ and a substair Γ' of L which satisfies the hypotheses of Proposition 1.3.13.

Proof. (of Proposition 1.3.13). We start by introducing some notation.

Let $\mathscr{F}, \mathscr{F}'$ be two G-constellations. Given a proper subrepresentation $\mathscr{E} < \mathscr{F}$ (resp. $\mathscr{E}' < \mathscr{F}'$) we will denote by $\mathscr{E}$ (resp. $\mathscr{E}'$) the corresponding subrepresentation $\mathscr{E}' < \mathscr{F}'$ (resp. $\mathscr{E} < \mathscr{F}$). Here, by "corresponding" we mean that, since $\mathscr{E}$ is a subrepresentation of the regular representation $\mathbb{C}[G]$ of an abelian group, it decomposes as a direct sum of distinct irreducible representations $\mathscr{E} \cong \bigoplus_j \rho_{i_j}$. We will denote by $\mathscr{E}'$ the subrepresentation of $\mathscr{F}' \cong \mathbb{C}[G]$ given by the same summands:

$$\mathscr{E}' \cong \bigoplus_j \rho_{i_j}.$$

In particular, for all $\theta \in \Theta$, the two rational numbers

$$\theta(\mathscr{E}) \text{ and } \theta(\mathscr{E}')$$

are the same. Moreover, we will denote by $\Gamma_{\mathscr{E}} \subset \Gamma$ (resp. $\Gamma_{\mathscr{E}'} \subset \Gamma'$) the substair associated to $\mathscr{E}$ (resp. $\mathscr{E}'$).

Notice that, given a proper G-equivariant $\mathbb{C}[x, y]$-submodule $\mathscr{E} < \mathscr{F}$, the subrepresentation $\mathscr{E}'$ is not necessarily a $\mathbb{C}[x, y]$-submodule of $\mathscr{F}'$. We are now ready to proceed with the proof.

(2)⇔*(3)* We omit the easy proof.

(1)⇒*(3)* Suppose, by contradiction, that $\Gamma' \subset L$ has a vertical left cut. Then, by Remark 1.3.8, the subrepresentation $\mathscr{E}_{\Gamma\cap\Gamma'} < \mathscr{F}$ is a $\mathbb{C}[x, y]$-submodule because, in Γ, the substair $\Gamma \cap \Gamma'$ has

a vertical left cut by hypothesis and its last box is not internal. At the same time, again by Remark 1.3.8, $\mathscr{E}'_{\Gamma\cap\Gamma'} < \mathscr{F}'$ is the complement of a $\mathbb{C}[x,y]$-submodule, because its first box is not internal and it has a vertical right cut. Hence,

$$C \subset \Theta_{\mathscr{F}} \cap \Theta_{\mathscr{F}'} \subset \{\theta(\mathscr{E}_{\Gamma\cap\Gamma'}) > 0\} \cap \{-\theta(\mathscr{E}_{\Gamma\cap\Gamma'}) > 0\} = \emptyset,$$

which contradicts (*1*).

(*3*)⇒(*1*) In order to prove statement (*1*), we need to show that

$$\Theta_{\mathscr{F}} \cap \Theta_{\mathscr{F}'} \neq \emptyset.$$

We start by identifying the proper irreducible G-equivariant subsheaves $\mathscr{E} < \mathscr{F}$ (resp. $\mathscr{E}' < \mathscr{F}'$) such that also $\mathscr{E}'$ (resp. $\mathscr{E}$) is a proper G-equivariant subsheaf of $\mathscr{F}$ (resp. $\mathscr{F}'$).

Let $\mathscr{E}' < \mathscr{F}'$ be a proper irreducible G-equivariant submodule of $\mathscr{F}'$; we consider three different cases.

Case 1. Both the first and the last box of the substair $\Gamma_{\mathscr{E}'} \subset \Gamma'$ are internal endpoints. Then, the same happens for $\Gamma_{\mathscr{E}} \subset \Gamma$. This is true because Γ has a vertical right cut in L, by the construction of a decreasing linking stair (see Definition 1.3.11), and hence, the right internal endpoint of $\Gamma_{\mathscr{E}'}$ in Γ', which is a horizontal cut by Remark 1.3.8, is different from the right internal endpoint of Γ in L. Therefore, both internal endpoints of $\Gamma_{\mathscr{E}'}$ correspond to internal endpoints of $\Gamma_{\mathscr{E}}$ of the same respective nature. As a consequence, the subrepresentation $\mathscr{E}$ is a proper, non necessarily irreducible, G-equivariant submodule of $\mathscr{F}$.

Case 2. The substair $\Gamma_{\mathscr{E}'}$ has only the vertical left cut in Γ', and hence, its last box coincides with the last box of Γ'. In particular, this box is not the right internal endpoint of Γ in L. We have to study the nature of the internal endpoints of $\Gamma_{\mathscr{E}}$. Notice first that it is enough to study the right internal endpoint of $\Gamma_{\mathscr{E}}$ because, if $\Gamma_{\mathscr{E}}$ has still left internal endpoint, then it is a vertical left cut. Let ρ_i be the label on the last box of Γ', then, the label on the horizontal left cut of Γ' (i.e. its first box) is ρ_{i+1}. Now, since, by hypothesis (*3*), the box labeled by ρ_{i+1} is a horizontal left cut of $\Gamma' \subset L$, the box labeled by ρ_i in Γ has to be a horizontal right cut for the substair $\Gamma_{\mathscr{E}}$. Therefore, $\Gamma_{\mathscr{E}}$ has only vertical left cuts and horizontal right cuts, and so, by Remark 1.3.8, $\mathscr{E}$ is a proper, non necessarily irreducible, G-equivariant submodule.

Case 3. The substair $\Gamma_{\mathscr{E}'} \subset \Gamma'$ has only the horizontal right cut, i.e. its first box coincides with the first box of Γ'. First of all notice that, as for the first analyzed case, the right internal endpoint of $\Gamma_{\mathscr{E}'}$ in Γ', which is a horizontal cut by hypothesis, is different from the right internal endpoint of Γ in L, which is vertical by definition of decreasing linking stair. Therefore, the box of Γ with the same label as the horizontal right cut of $\Gamma_{\mathscr{E}'}$ is an internal endpoint of $\Gamma_{\mathscr{E}}$ and it is a horizontal right cut. Finally, the first box of Γ' in L is a left internal endpoint for $\Gamma_{\mathscr{E}}$, and so it is a horizontal left cut by point (*3*) of the statement. As a consequence, $\Gamma_{\mathscr{E}}$ has two horizontal cuts.

In summary, if $\mathscr{E}' < \mathscr{F}'$ is a proper irreducible G-equivariant submodule of $\mathscr{F}'$ such that $\Gamma_{\mathscr{E}'}$ has a vertical left cut, then also $\mathscr{E} < \mathscr{F}$ is a proper, non necessarily irreducible,

G-equivariant submodule. While, if $\Gamma_{\mathscr{E}'} < \Gamma'$ has only the right horizontal cut, then $\Gamma_{\mathscr{E}}$ has two horizontal cuts.

Following the same logic, if $\mathscr{E} < \mathscr{F}$ is a proper irreducible G-equivariant submodule of $\mathscr{F}$ such that $\Gamma_{\mathscr{E}}$ has a horizontal right cut, then also $\mathscr{E}' < \mathscr{F}'$ is a proper, non necessarily irreducible, G-equivariant submodule. While, if $\Gamma_{\mathscr{E}} < \Gamma$ has only the left vertical cut, then $\Gamma_{\mathscr{E}'}$ has two vertical cuts.

We are now ready to exhibit a $\theta \in \Theta^{\mathrm{gen}}$ such $\mathscr{F}$ and $\mathscr{F}'$ are θ-stable. Let $\theta_{\mathscr{F}}$ and $\theta_{\mathscr{F}'}$ be the respective favorite conditions for $\mathscr{F}$ and $\mathscr{F}'$ and let $\theta = \theta_{\mathscr{F}} + \theta_{\mathscr{F}'}$ be their sum. Then, both $\mathscr{F}$ and $\mathscr{F}'$ are θ-stable. Indeed,

- if $\mathscr{E} < \mathscr{F}$ is a proper irreducible G-equivariant $\mathbb{C}[x,y]$-submodule of $\mathscr{F}$ such that also $\mathscr{E}'$ is a $\mathbb{C}[x,y]$-submodule of $\mathscr{F}'$, then
$$\theta(\mathscr{E}) = \theta_{\mathscr{F}}(\mathscr{E}) + \theta_{\mathscr{F}'}(\mathscr{E}) = \theta_{\mathscr{F}}(\mathscr{E}) + \theta_{\mathscr{F}'}(\mathscr{E}') > 0$$
follows from the fact that $\mathscr{F}$ is $\theta_{\mathscr{F}}$-stable and $\mathscr{F}'$ is $\theta_{\mathscr{F}'}$-stable (see Remark 1.3.10);
- if $\mathscr{E}' < \mathscr{F}'$ is a proper irreducible G-equivariant $\mathbb{C}[x,y]$-submodule of $\mathscr{F}'$ such that $\Gamma_{\mathscr{E}}$ has two horizontal cuts, then
$$\theta(\mathscr{E}') = \theta_{\mathscr{F}}(\mathscr{E}') + \theta_{\mathscr{F}'}(\mathscr{E}') = \theta_{\mathscr{F}}(\mathscr{E}) + \theta_{\mathscr{F}'}(\mathscr{E}') = \theta_{\mathscr{F}'}(\mathscr{E}') = 1 > 0$$
follows from the fact that $\mathscr{F}'$ is $\theta_{\mathscr{F}'}$-stable (see Remark 1.3.10) and from Remarks 1.3.8 and 1.3.9;
- if $\mathscr{E} < \mathscr{F}$ is a proper irreducible G-equivariant $\mathbb{C}[x,y]$-submodule of $\mathscr{F}$ such that $\Gamma_{\mathscr{E}'}$ has two vertical cuts, then
$$\theta(\mathscr{E}) = \theta_{\mathscr{F}}(\mathscr{E}) + \theta_{\mathscr{F}'}(\mathscr{E}) = \theta_{\mathscr{F}}(\mathscr{E}) + \theta_{\mathscr{F}'}(\mathscr{E}') = \theta_{\mathscr{F}}(\mathscr{E}) = 1 > 0$$
follows from the fact that $\mathscr{F}$ is $\theta_{\mathscr{F}}$-stable (see Remark 1.3.10) and from Remarks 1.3.8 and 1.3.9;
- if $\mathscr{E} < \mathscr{F}$ (resp. $\mathscr{E}' < \mathscr{F}'$) is a proper reducible G-equivariant $\mathbb{C}[x,y]$-submodule, then
$$\theta(\mathscr{E}) > 0$$
follows by applying the previous points to the irreducible components of $\mathscr{E}$ and from the additivity of θ.

The last issue here is that, in general, such θ is not generic, i.e.
$$\theta \in \overline{\Theta_{\mathscr{F}} \cap \Theta_{\mathscr{F}'}} \smallsetminus \Theta_{\mathscr{F}} \cap \Theta_{\mathscr{F}'}.$$

In order to solve this problem, we can perturb $\theta_{\mathscr{F}}$ and $\theta_{\mathscr{F}'}$ the same way as as we did in Remark 1.3.10 thus obtaining a generic $\widetilde{\theta} \in \Theta_{\mathscr{F}} \cap \Theta_{\mathscr{F}'}$. Consider the stability conditions

$\varepsilon, \varepsilon' \in \Theta$ defined as follows:

$$\begin{cases} \varepsilon_i = 0 & \text{if } \rho_i \text{ is an antigenerator of } \Gamma_{\mathcal{F}}, \\ \varepsilon'_i = 0 & \text{if } \rho_i \text{ is an antigenerator of } \Gamma_{\mathcal{F}'}, \\ \varepsilon_i < 0 & \text{if } \rho_i \text{ is neither a generator nor an antigenerator of } \Gamma_{\mathcal{F}}, \\ \varepsilon'_i < 0 & \text{if } \rho_i \text{ is neither a generator nor an antigenerator of } \Gamma_{\mathcal{F}'}, \\ \varepsilon_i = -\sum\limits_{\rho_j \in (\Gamma_{\rho_i} \smallsetminus \rho_i)} \varepsilon_j & \text{if } \rho_i \text{ is a generator of } \Gamma_{\mathcal{F}}, \\ \varepsilon'_i = -\sum\limits_{\rho_j \in (\Gamma'_{\rho_i} \smallsetminus \rho_i)} \varepsilon'_j & \text{if } \rho_i \text{ is a generator of } \Gamma_{\mathcal{F}'}, \\ \sum\limits_{\rho_i \text{ generator of } \Gamma_{\mathcal{F}}} \varepsilon_i + \sum\limits_{\rho_i \text{ generator of } \Gamma_{\mathcal{F}'}} \varepsilon'_i < 1, \end{cases}$$

where, as in Remark 1.3.10, $\Gamma_{\rho_i} \subset \Gamma$ (resp. $\Gamma'_{\rho_i} \subset \Gamma'$) is the substair associated to the $\mathbb{C}[x,y]$-submodule of $\mathcal{F}$ (resp. $\mathcal{F}'$) generated by the irreducible subrepresentation ρ_i.

Now, if

$$\widetilde{\theta} = (\theta_{\mathcal{F}} + \varepsilon) + (\theta_{\mathcal{F}'} + \varepsilon')$$

then $\mathcal{F}$ and $\mathcal{F}'$ are $\widetilde{\theta}$-stable, and ε and ε' can be chosen in such a way that $\widetilde{\theta}$ is generic. As a consequence $\Theta_{\mathcal{F}} \cap \Theta_{\mathcal{F}'} \neq \emptyset$.

□

We will see, in the proof of Theorem 1.3.17, that there is an easier way to prove that $\Theta_{\mathcal{F}} \cap \Theta_{\mathcal{F}'}$ is not empty. By following the same logic, one can prove a similar statement for the increasing linking stairs.

Proposition 1.3.15. *Let Γ be the abstract G-stair of a G-constellation $\mathcal{F}$ and let L be its abstract increasing linking stair. Consider any G-stair $\Gamma' \subset L$ and its associated G-constellation $\mathcal{F}'$. Then, the following are equivalent:*

1. *there exists at least a chamber C such that both $\mathcal{F}$ and $\mathcal{F}'$ belong to C, i.e. $\Theta_{\mathcal{F}} \cap \Theta_{\mathcal{F}'} \neq \emptyset$,*
2. $\mathfrak{h}(\mathcal{F}') = \mathfrak{h}(\mathcal{F}) + 1$,
3. *the substair $\Gamma' \subset L$ has a right vertical cut.*

In particular, $\mathcal{F}$ is the G-constellation next to $\mathcal{F}'$ in $\mathcal{M}_C$ in the sense of Remark 1.3.1.

1.3.3 Counting the chambers

Remark 1.3.16. Propositions 1.3.13 and 1.3.15 provide a way to build 1-dimensional families of nilpotent G-constellations. In particular, each of this families corresponds to some exceptional line in some $\mathcal{M}_C$. Moreover, the two gluings described in the definition of linking stair are nothing but the two possible ways of deforming a toric G-constellation keeping the property of being nilpotent described in Remark 1.3.2. This implies that the families coming from Proposition 1.3.13 and Proposition 1.3.15 are exactly the 1-dimensional families of nilpotent G-constellations appearing in the moduli spaces $\mathcal{M}_C$.

An easy combinatorial computation tells us that the maximum number of chambers is $k!$. Indeed, if we start by a G-constellation $\mathscr{F}_1$ of maximum height $\mathfrak{h}(\mathscr{F}) = k$, i.e. $\mathscr{F}_1$ has one of the k abstract G-stairs shown in Figure 1.13, we can construct irreducible toric G-

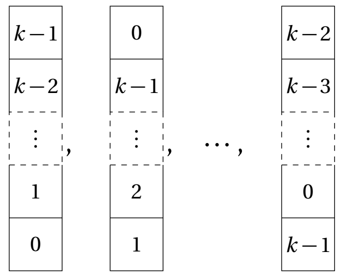


Figure 1.13. The abstract G-stairs of maximum height.

constellations $\mathscr{F}_2,\dots,\mathscr{F}_k$ with respective abstract G-stairs Γ_j for $j = 2,\dots,k$ by recursively applying the prescriptions in Proposition 1.3.13. Precisely, for any $j > 1$, each Γ_j is a connected substair, with horizontal left cut, of the decreasing linking stair of Γ_{j-1}. This is true because at each step the number of possible horizontal left cuts in the decreasing linking stairs decreases by one.

To conclude that the maximum number of chambers is $k!$, we notice that the j-th time that we apply Proposition 1.3.13 there are $k-j$ possible G-stairs with horizontal left cut in the decreasing linking stair of the abstract G-stair of $\mathscr{F}_j$.

Theorem 1.3.17. *If $G \subset \mathrm{SL}(2,\mathbb{C})$ is a finite abelian subgroup of cardinality $k = |G|$, then the space of generic stability conditions Θ^{gen} is the disjoint union of $k!$ chambers.*

Proof. It is enough to show that, if $\mathscr{F}_1,\dots,\mathscr{F}_k$ are as in Remark 1.3.16, then there exists a chamber

$$C = \Theta_{\mathscr{F}_1} \cap \Theta_{\mathscr{F}_2} \cap \cdots \cap \Theta_{\mathscr{F}_k} \neq 0,$$

such that $\mathscr{F}_j$ is C-stable for all $j = 1,\dots,k$. We claim that, if, for all $j = 1,\dots,k$, the favorite condition of $\mathscr{F}_j$ is $\theta_{\mathscr{F}_j}$, then

$$\theta = \sum_{j=1}^{k} \theta_{\mathscr{F}_j} \in C.$$

A priori, in order to prove the claim, we need to show both that θ is generic and that every $\mathscr{F}_j$ is θ-stable. In fact, it is enough to show just that every $\mathscr{F}_j$ is θ-stable, because this implies that $\mathscr{M}_\theta$ has k torus fixed-points and, as a consequence, that θ is generic.

Let $\mathscr{E}_j < \mathscr{F}_j$ be a proper G-equivariant irreducible $\mathbb{C}[x,y]$-submodule of $\mathscr{F}_j$ with substair $\Gamma_{\mathscr{E}_j} \subset \Gamma_{\mathscr{F}_j}$. Suppose also that $\mathscr{E}_j = \bigoplus_{s=m}^{n} \rho_s$, where $0 \le m \le n \le k-1$. We will denote by $\mathscr{E}_i$, for $i = 1,\dots,j-1,j+1,\dots,k$, the subrepresentation of $\mathscr{F}_i$ corresponding to $\mathscr{E}_j$, i.e.

$$\mathscr{E}_i = \bigoplus_{s=m}^{n} \rho_s, \quad \forall i = 1,\dots,j-1,j+1,\dots,k.$$

Notice that

- if $\Gamma_{\mathscr{E}_{j+1}}$ has two vertical cuts, then $\Gamma_{\mathscr{E}_i}$ has two vertical cuts for every $i > j+1$;

- if $\Gamma_{\mathscr{E}_{j-1}}$ has two horizontal cuts, then $\Gamma_{\mathscr{E}_i}$ has two horizontal cuts for every $i < j-1$.

This is true because every time we increase (resp. decrease) the index i, we perform a horizontal left (resp. vertical right) cut in the decreasing (resp. increasing) linking stair which does not affect the vertical left (resp. horizontal right) cut of $\Gamma_{\mathscr{E}_{j+1}}$ (resp. $\Gamma_{\mathscr{E}_{j-1}}$).

Hence, for all $i = 1,\dots, j-1, j+1,\dots,k$, we have $\theta_{\mathscr{F}_i}(\mathscr{E}_j) \geq 0$ and, as a consequence

$$\theta(\mathscr{E}_j) = \left(\theta_{\mathscr{F}_j} + \sum_{i\neq j} \theta_{\mathscr{F}_i}\right)(\mathscr{E}_j) > 0.$$

□

Remark 1.3.18. The proof of Theorem 1.3.17 provides an alternative way to prove that

$$\Theta_{\mathscr{F}} \neq \emptyset$$

in Remark 1.3.10 and, that

$$\Theta_{\mathscr{F}} \cap \Theta_{\mathscr{F}'} \neq \emptyset$$

in the last part of the third point of the proof of Proposition 1.3.13.

For example, let $\mathscr{F}$ be a toric G-constellation with abstract G-stair of height $\mathfrak{h}(\mathscr{F}) = j$. We construct $\mathscr{F}_1,\dots,\mathscr{F}_{j-1},\mathscr{F}_{j+1},\dots,\mathscr{F}_k$ by recursively applying Propositions 1.3.13 and 1.3.15, i.e.

- if $i > j$, then $\mathscr{F}_i$ has, as G-stair, a G-substair, with a horizontal left cut, of the decreasing linking stair of $\mathscr{F}_{i-1}$,
- if $i < j$, then $\mathscr{F}_i$ has, as G-stair, a G-substair, with a vertical right cut, of the increasing linking stair of $\mathscr{F}_{i+1}$.

Then, if $\theta = \theta_{\mathscr{F}} + \sum \theta_{\mathscr{F}_i}$ is the sum of all favorite conditions, we have $\theta \in \Theta_{\mathscr{F}}$.

1.4 Simple chambers

In this section I will firstly introduce the notion of chamber stair which is a stair that encodes all the data needed to reconstruct a chamber. Then, I will define simple chambers, which are a particular kind of chambers with the property that any toric G-constellation belongs to at least one of them. Finally, I will prove that there are exactly $k \cdot 2^{k-2}$ simple chambers.

Remark 1.4.1. Given a chamber $C \subseteq \Theta^{\text{gen}}$ we can make a stair out of it, and we will call it the chamber stair.

Let $\mathscr{F}_1, \dots, \mathscr{F}_k$ be the toric G-constellations in $\mathscr{M}_C$. As explained in Proposition 1.3.13 (resp. Proposition 1.3.15), the abstract G-stairs Γ_j, Γ_{j+1} of two consecutive G-constellations $\mathscr{F}_j, \mathscr{F}_{j+1}$ are substairs of the same stair L, namely the decreasing linking stair of Γ_j (resp. the increasing linking stair of Γ_{j+1}). Moreover they have non-empty intersection in L.

Now, if $\Gamma_1, \dots, \Gamma_k$ are the respective abstract G-stairs of $\mathscr{F}_1, \dots, \mathscr{F}_k$, we can construct a new abstract stair Γ_C by gluing consecutive abstract G-stairs along their common parts.

Definition 1.4.2. The *abstract chamber stair of C* or the *abstract C-stair* is the abstract stair Γ_C obtained as described above.

Example 1.4.3. Consider the case $G \cong \mathbb{Z}/5\mathbb{Z}$. Figure 1.14 explains how to build an abstract C-stair starting from the abstract G-stairs of the G-constellations in some chamber C.

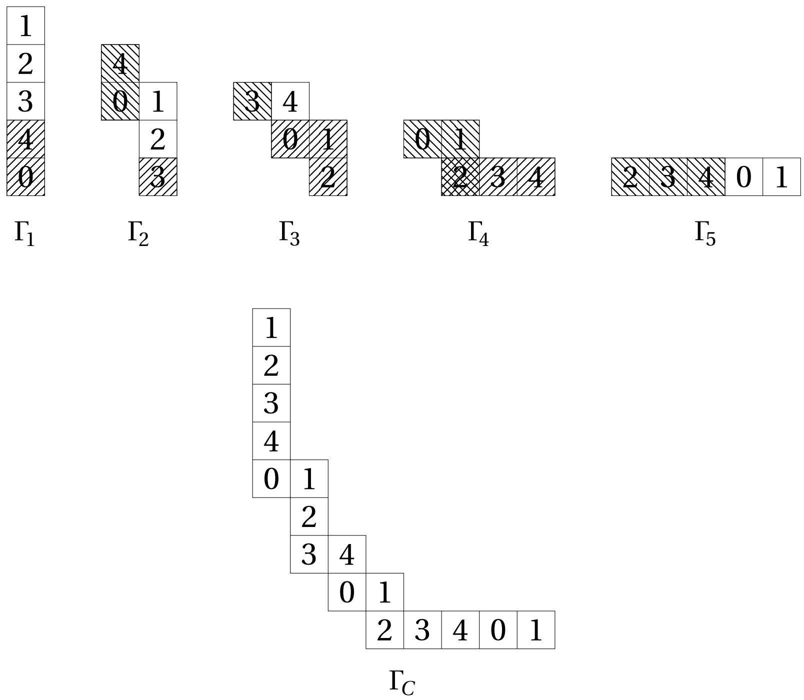


Figure 1.14. The abstract C-stair Γ_C is obtained by gluing, along their common part, the abstract $\mathbb{Z}/5\mathbb{Z}$-stairs Γ_i and Γ_{i+1} for $i = 1, \dots, 4$.

In particular, we have glued the boxes of an abstract G-stair with the boxes of the next abstract G-stair.

Definition 1.4.4. A *chamber stair associated to C* or a *C-stair* is any realisation $\widetilde{\Gamma}_C$ of the abstract chamber stair Γ_C associated to C as a subset of the representation tableau.

Remark 1.4.5. Let $C \subset \Theta^{\text{gen}}$ be a chamber and let $\Gamma_C \subset \mathscr{T}_G$ be a C-stair. Consider a G-stair $\Gamma \subset \Gamma_C$ of width $\mathfrak{w}(\Gamma) = j$ and the associated G-constellation $\mathscr{F}_\Gamma$. Let us also denote by $b, b' \in \Gamma$ the first and the last box of Γ. Suppose that $\mathscr{F}_\Gamma$ is not C-stable. Then, there are two consecutive C-stable G-constellations $\mathscr{F}$ and $\mathscr{F}'$ with associated respective G-stairs $\Gamma_{\mathscr{F}}, \Gamma_{\mathscr{F}'} \subset \Gamma_C$ such that $b \in \Gamma_{\mathscr{F}}$ and $b' \in \Gamma_{\mathscr{F}'}$.

Therefore, Γ is a substair of both the decreasing linking stair L of $\Gamma_{\mathscr{F}}$ and the increasing linking stair L' of $\Gamma_{\mathscr{F}'}$. In particular, as a consequence of Proposition 1.3.13 (and of Proposition 1.3.15), one and only one between the following two possibilities must occur, namely:

(1.4.1)
$$\begin{array}{l} \mathfrak{w}(\mathscr{F}) = j-1, \mathfrak{w}(\mathscr{F}') = j, \text{ and } b \text{ (resp. } b'\text{) is a left (resp. right) horizontal cut of } \Gamma \text{ in } L, \\ \mathfrak{w}(\mathscr{F}) = j, \mathfrak{w}(\mathscr{F}') = j+1, \text{ and } b \text{ (resp. } b'\text{) is a right (resp. left) vertical cut of } \Gamma \text{ in } L. \end{array}$$

On the other hand, again as a consequence of Proposition 1.3.13 and Proposition 1.3.15, if $\mathscr{F}_\Gamma$ is C-stable, none of the conditions in 1.4.1 can hold true, and, in this case, Γ has horizontal left cut and vertical right cut in Γ_C.

Summing up, if $\Gamma \subset \Gamma_C$ is a connected G-substair associated to a toric G-constellation $\mathscr{F}_\Gamma$ then only the following two cases can occur:

- the G-constellation $\mathscr{F}_\Gamma$ is C-stable and Γ has a horizontal left cut and a vertical right cut, or
- the G-constellation $\mathscr{F}_\Gamma$ is not C-stable and Γ has two horizontal cuts or two vertical cuts.

Lemma 1.4.6. *Different chambers have different abstract chamber stairs.*

Proof. First, recall from Remark 1.4.5 that, as per Proposition 1.3.13, the G-stair of any toric C-stable G-constellation has a vertical right cut in the C-stair and a horizontal right cut in the decreasing linking stair of the previous G-constellation.

Suppose that two chambers C and C' have the same abstract chamber stair Γ. In particular, from the construction of abstract chamber stairs, it follows that C and C' have the same first (in the sense of Remark 1.3.1) toric G-constellation. Suppose that C and C' differ for the j-th toric G-constellation. This translates into the fact that, if $\mathscr{F}_j$ and $\mathscr{F}'_j$ are the respective j-th G-constellations of C and C' and Γ_j, Γ'_j are their abstract G-stairs, then

$$\Gamma_j \neq \Gamma'_j.$$

Notice that, calling $\mathscr{F}_{j-1}$ the $(j-1)$-th toric G-constellation of C (and C') and calling Γ_{j-1} its abstract G-stair, both Γ_j and Γ'_j are substairs of the decreasing linking stair L_{j-1} of Γ_{j-1} and they have horizontal right cut in L_{j-1} as noticed above. Since, Γ_{j-1}, Γ_j and Γ'_j are connected and $\Gamma_{j-1} \cap \Gamma_j, \Gamma_{j-1} \cap \Gamma'_j \neq \emptyset$ in L_{j-1}, it follows that:

$$\Gamma_{j-1} \cup \Gamma_j \subsetneq \Gamma_{j-1} \cup \Gamma'_j \text{ or } \Gamma_{j-1} \cup \Gamma_j \supsetneq \Gamma_{j-1} \cup \Gamma'_j.$$

Finally, if, without loss of generality, we suppose

$$\Gamma_{j-1} \cup \Gamma_j \subsetneq \Gamma_{j-1} \cup \Gamma'_j \subset \Gamma,$$

we get a contradiction. Indeed, as noticed at the beginning, Γ_j has a vertical right cut in Γ, but it has to have a horizontal right cut in $\Gamma_{j-1} \cup \Gamma'_j$ because it is a connected substair of L_{j-1} which strictly contains Γ_j. □

Remark 1.4.7. Since the abstract chamber stair Γ_C of a chamber C contains a copy of the abstract G-stairs of the toric C-stable G-constellations, we will think of such abstract G-stairs as substairs of Γ_C.

Similarly, given a C-stair $\widetilde{\Gamma}_C \subset \mathscr{T}_G$ which realize Γ_C, we will realize the abstract G-stairs associated to the G-constellations in C as substairs of $\widetilde{\Gamma}_C$.

Definition 1.4.8. Given a chamber C, we will say that a toric C-stable G-constellation is C*-characteristic* if its abstract G-stair has the same generators as the abstract C-stair.

We will say that a chamber C is *simple* if there is a toric C-stable G-constellation whose abstract G-stair has the same generators of the abstract C-stair, i.e. if there exists at least one C-characteristic G-constellation.

Example 1.4.9. An example of a simple chamber is given by the chamber C_G in Theorem 1.0.21, i.e. the chamber whose associated moduli space is G-Hilb($\mathbb{A}^2$). In particular, the abstract C_G-stair has only one generator, namely ρ_0.

Definition 1.4.10. Let Γ be a G-stair and let ρ_i and ρ_j be its first and its last generators.

- We will call *left tail* of Γ the substair of Γ given by

$$\mathfrak{lt}(\Gamma) = \left\{ y^s \cdot \rho_i \,\middle|\, s > 0 \right\}.$$

- We will call *right tail* of Γ the substair of Γ given by

$$\mathfrak{rt}(\Gamma) = \left\{ x^s \cdot \rho_j \,\middle|\, s > 0 \right\}.$$

- We will call *tail* of Γ the substair of Γ given by

$$\mathfrak{t}(\Gamma) = \mathfrak{lt}(\Gamma) \cup \mathfrak{rt}(\Gamma).$$

Similarly one can define left/right tails for abstract G-stairs.

Remark 1.4.11. If two G-stairs Γ and Γ' have the same generators, then they differ by their tails, i.e. the following equality of subsets of the representation tableau holds true:

$$\Gamma \setminus \mathfrak{t}(\Gamma) = \Gamma' \setminus \mathfrak{t}(\Gamma')$$

In particular, if a G-stair Γ has a tail of cardinality m, then there are $m+1$ G-stairs with the same generators as Γ.

In simple words, the other G-stairs are obtained by moving some boxes from the left tail to the right tail (and viceversa) of Γ.

Proposition 1.4.12. *The following properties are true.*

1. *Any toric G-constellation is C-stable for some simple chamber C.*

2. *In order to find all the toric C-stable G-constellations of a simple chamber C, it is enough to know at least one C-characteristic G-constellation.*

3. *If C is a simple chamber, all the toric G-constellations that admit a G-stair with the same generators as the C-stair belong to C, i.e. they are C-stable. In particular, they are C-characteristic.*

Proof. Let Γ_C be the abstract C-stair. We will prove the first two points in a constructive way. In order to do so, we will show that, given a toric G-constellation $\mathscr{F}$, there is a unique simple chamber C such that $\mathscr{F}$ is C-characteristic.

Let $\mathscr{F}$ be a toric G-constellation with associated abstract G-stair $\Gamma_{\mathscr{F}}$ of height $\mathfrak{h}(\mathscr{F}) = j$. In order to build a chamber starting from $\mathscr{F}$, we have to first recursively apply Propositions 1.3.13 and 1.3.15 $j-1$ times and $k-j$ times respectively, to obtain k toric constellations

$$\mathscr{F}_1, \dots, \mathscr{F}_{j-1}, \mathscr{F}, \mathscr{F}_{j+1}, \dots, \mathscr{F}_k$$

and, finally, apply Theorem 1.3.17 to conclude that there exists a chamber C such that the constellations $\mathscr{F}_1, \dots, \mathscr{F}_{j-1}, \mathscr{F}, \mathscr{F}_{j+1}, \dots, \mathscr{F}_k$ correspond to the toric points of $\mathscr{M}_C$.

The condition that the chamber must be simple translates into the fact that, at every step, no new generators appear. This may be only achieved by performing, every time that we apply Proposition 1.3.13 (resp. Proposition 1.3.15), the first (resp. last) possible horizontal (resp. vertical) cut in the decreasing (resp. increasing) linking stair.

In order to prove the last point, we start by considering a G-constellation $\mathscr{F}$ whose abstract G-stair $\Gamma_{\mathscr{F}}$ has the same generators as the C-stair and such that it has empty right tail, i.e. $\mathfrak{t}(\Gamma_{\mathscr{F}}) = \mathfrak{lt}(\Gamma_{\mathscr{F}})$.

Let $m = \#\mathfrak{lt}(\Gamma_{\mathscr{F}})$ be the cardinality of the left tail of $\Gamma_{\mathscr{F}}$. The first m times we apply Proposition 1.3.13 by performing the first possible horizontal cut we increase the cardinality of $\mathfrak{rt}(\Gamma_{\mathscr{F}})$ by 1 and, consequently, we decrease the cardinality of $\mathfrak{lt}(\Gamma_{\mathscr{F}})$ by 1. In this way we find, as explained in Remark 1.4.11, all the toric G-constellations which admit a G-stair with the same generators as the C-stair and all of them are C-stable by Theorem 1.3.17. □

Lemma 1.4.13. *Let Γ be a G-stair. Then Γ has at most*

$$\left\lfloor \frac{k+1}{2} \right\rfloor$$

generators.

Proof. The statement follows from the following observation. If a stair has r generators, then it has at least $2r-1$ boxes, as shown in Figure 1.15.

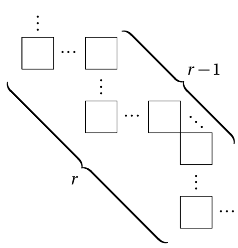


Figure 1.15.

Now, a G-stair has exactly k boxes. Hence,

$$r \leq \left\lfloor \frac{k+1}{2} \right\rfloor.$$

$\square$

Example 1.4.14. Non-simple chambers exist.

As already mentioned in Theorem 1.0.21, there is a chamber C_G such that $G\text{-Hilb}(\mathbb{A}^2) \cong \mathscr{M}_{C_G}$ as moduli spaces. In particular,

$$C_G \subset \{\, \theta \in \Theta \mid \theta_0 < 0,\ \theta_i > 0\ \forall i = 1, \dots, k-1 \,\},$$

and the abstract G-stairs of its toric constellations are shown in Figure 1.16.

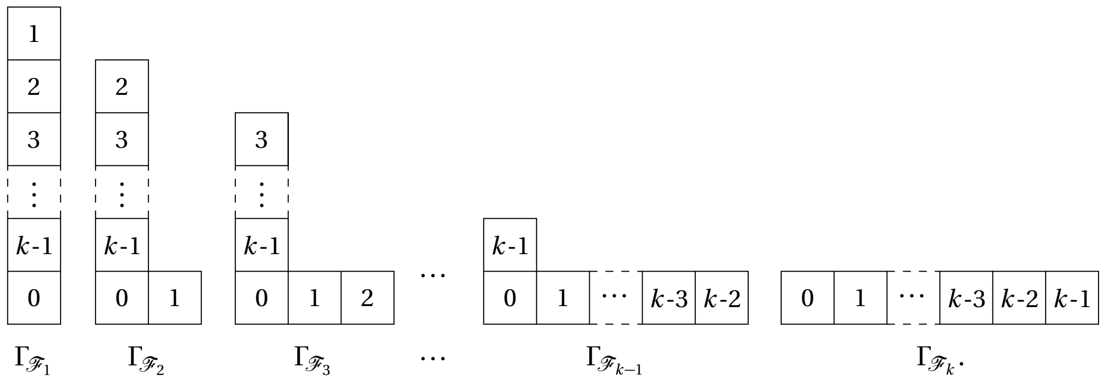


Figure 1.16. The abstract G-stairs of the C_G-stable toric G-constellations.

Notice that, for $i = 1, \dots, k$ and $j = 0, \dots, k-1$, the favorite conditions $\theta_{\mathscr{F}_i}$ are defined by

$$(\theta_{\mathscr{F}_i})_j = \begin{cases} -2 & \text{if } j = 0 \,\&\, i \neq 1, k, \\ -1 & \text{if } j = 0 \,\&\, (i = 1 \text{ or } i = k), \\ 1 & \text{if } j = i-1 \neq 0, \\ 1 & \text{if } j = i, \\ 0 & \text{otherwise.} \end{cases}$$

and that the condition

$$\theta = \sum_{i=1}^{k} \theta_{\mathscr{F}_i} = (-2k+2, \underbrace{2, \dots, 2}_{k-1})$$

belongs to C_G. More precisely, the moduli space $G\text{-Hilb}(\mathbb{A}^2)$ parametrises all the toric G-constellations generated by the trivial representation. As a consequence, the abstract G-stairs $\Gamma_{\mathscr{F}_i}$, for $i = 1, \dots, k$, have, as only generator, the trivial representation.

Let us reverse this property by asking the presence of just one antigenerator, for example, the trivial representation. It is easy to see that there is a chamber C_G^{OP} whose toric G-constellations, as requested, have the abstract G-stairs in Figure 1.17.

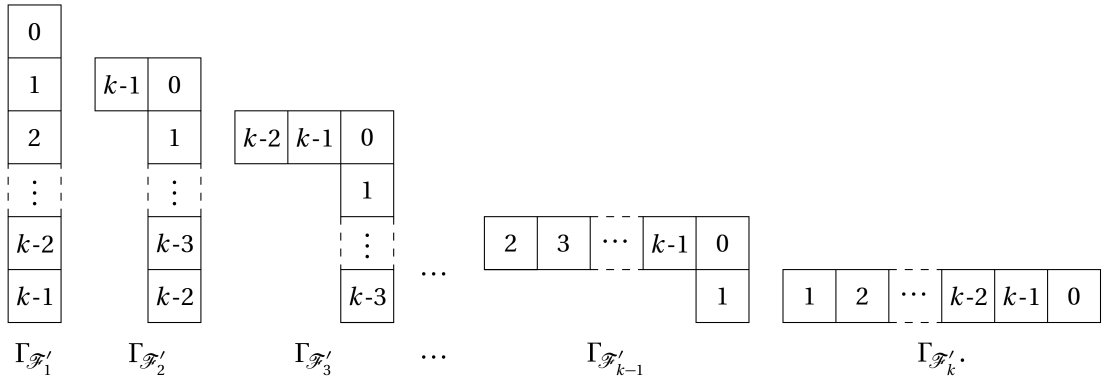


Figure 1.17. The abstract G-stairs of the C_G^{OP}-stable toric G-constellations.

In particular,

$$C_G^{\mathrm{OP}} \subset \{\, \theta \in \Theta \mid \theta_0 > 0,\ \theta_i < 0\ \forall i = 1, \ldots, k-1 \,\}.$$

We will call the associated moduli space

$$G\text{-}\mathrm{Hilb}^{\mathrm{OP}}(\mathbb{A}^2) := \mathcal{M}_{C_G^{\mathrm{OP}}}.$$

Notice that, while $C^{\mathrm{OP}}_{\mathbb{Z}/3\mathbb{Z}}$ is simple, $C^{\mathrm{OP}}_{\mathbb{Z}/k\mathbb{Z}}$ is not simple for $k \geq 4$ because the number of generators of the $C^{\mathrm{OP}}_{\mathbb{Z}/k\mathbb{Z}}$-stair is

$$k-1 > \left\lfloor \frac{k+1}{2} \right\rfloor \quad \forall k \geq 4.$$

Therefore, as a consequence of Lemma 1.4.13, there is no $C^{\mathrm{OP}}_{\mathbb{Z}/k\mathbb{Z}}$-characteristic G-constellation.

We show, as an example, the abstract chamber stairs of C_G and C_G^{OP} in the case $k = 5$.

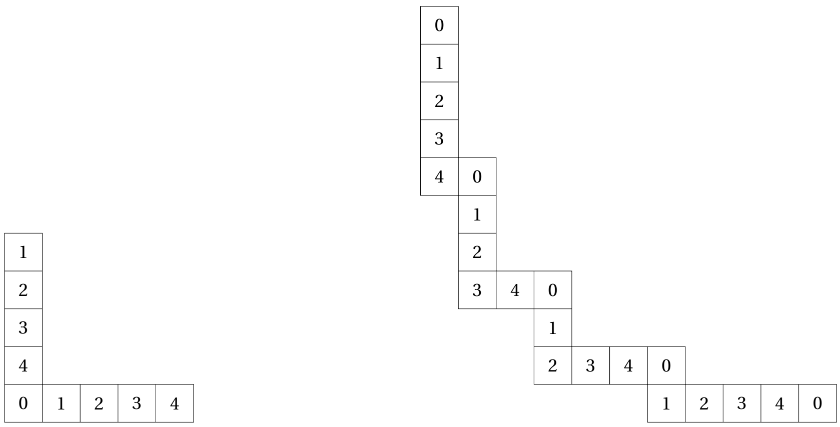


Figure 1.18. The abstract $C_{\mathbb{Z}/5\mathbb{Z}}$-stair and the abstract $C^{\mathrm{OP}}_{\mathbb{Z}/5\mathbb{Z}}$-stair.

Theorem 1.4.15. *If $G \subset \mathrm{SL}(2,\mathbb{C})$ is a finite abelian subgroup of cardinality $k = |G|$, then the space of generic stability conditions Θ^{gen} contains $k \cdot 2^{k-2}$ simple chambers.*

Proof. Let $\mathscr{B}$ be the set of of possible sets of generators for a G-stair, i.e.

$$\mathscr{B} = \left\{ A \subset \mathscr{T}_G \,\middle|\, \text{there exists a } G\text{-stair whose generators are the elements in } A \right\},$$

and let $\mathscr{G}$ be the set of all G-stairs

$$\mathscr{G} = \{\Gamma \subset \mathscr{T}_G \mid \Gamma \text{ is a } G\text{-stair}\}.$$

Consider the subsemigroup Z of $\mathscr{T}_G$

$$Z = \left\{(\alpha k + \gamma, \beta k + \gamma, \rho_0) \in \mathscr{T}_G \;\middle|\; \alpha, \beta, \gamma \geq 0\right\}.$$

We will denote by $\overline{\mathscr{B}}$ and $\overline{\mathscr{G}}$ the set of equivalence classes

$$\overline{\mathscr{B}} = \mathscr{B}/\sim_Z, \text{ and } \overline{\mathscr{G}} = \mathscr{G}/\sim_Z$$

where, if $A_1, A_2 \in \mathscr{B}$ (resp. $\Gamma_1, \Gamma_2 \in \mathscr{G}$), then $A_1 \sim_Z A_2$ (resp. $\Gamma_1 \sim_Z \Gamma_2$) if there exist $z \in Z$ such that

$$A_1 = A_2 + z \text{ or } A_2 = A_1 + z \quad (\text{resp. } \Gamma_1 = \Gamma_2 + z \text{ or } \Gamma_2 = \Gamma_1 + z).$$

Notice that, if two G-stairs are $\sim_Z$-equivalent also their sets of generators are $\sim_Z$-equivalent. However, the contrary is not true.

Now, the number of simple chambers equals the cardinality of $\overline{\mathscr{B}}$. Indeed, Proposition 1.4.12 implies that the chamber C is uniquely determined by a constellation $\mathscr{F}$ whose G-stair has the same generators as the C-stair. More precisely, C is uniquely determined by the generators of any characteristic C-stair $\Gamma_{\mathscr{F}}$. Although there are infinitely many G-stairs corresponding to $\mathscr{F}$, Remark 1.2.18 tells us that two G-stairs correspond to the same G-constellation if and only if they differ by an element in Z, i.e. they are $\sim_Z$-equivalent.

Let $\mathscr{G}_r$ be the set of G-stairs with r generators and let $\overline{\mathscr{G}}_r = \mathscr{G}_r/\sim_Z$ be the induced quotient. We have a surjective map

$$\Psi : \mathscr{G} \longrightarrow \mathscr{B}$$

which associates to each G-stair its set of generators, and this map descends to the sets of equivalence classes

$$\overline{\Psi} : \overline{\mathscr{G}} \longrightarrow \overline{\mathscr{B}},$$

because $\sim_Z$-equivalent G-stairs correspond to $\sim_Z$-equivalent sets of generators.

Now, $\overline{\mathscr{B}}$ decomposes as a disjoint union (see Lemma 1.4.13) as follows:

$$\overline{\mathscr{B}} = \bigsqcup_{r=1}^{\lfloor \frac{k+1}{2} \rfloor} \overline{\Psi}(\overline{\mathscr{G}}_r).$$

Our strategy is to compute $\overline{\Psi}(\overline{\mathscr{G}}_r)$ for every $1 \leq r \leq \left\lfloor \frac{k+1}{2} \right\rfloor$ and then sum over all r. For $r = 1$ we have $|\overline{\Psi}(\overline{\mathscr{G}}_1)| = k$. If we impose the presence of $r \geq 2$ generators and of a tail of cardinality j then there are

$$k \cdot \binom{k-2-j}{2r-3}$$

elements in $\overline{\Psi}(\overline{\mathscr{G}}_r)$ which comes from G-stairs with a tail of cardinality j. Indeed, as shown in Figure 1.19, we have $2r-1$ fixed boxes (generators and anti-generators), j boxes contained in the tails (dashed areas) and $k-2r+1-j$ boxes left to arrange in $2r-2$ places (dotted areas). The number of possible ways to arrange the boxes is computed via the stars and bars method[1].

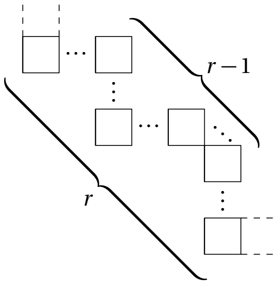


Figure 1.19.

In particular, there are

$$\binom{(2r-2)+(k-2r+1-j)-1}{k-2r+1-j} = \binom{k-2-j}{2r-3}$$

of them.

Finally, if we sum over all possible r and j, we get

$$k \cdot \left[1 + \sum_{r=2}^{\lfloor \frac{k+1}{2} \rfloor} \sum_{j=0}^{k-2r+1} \binom{k-2-j}{2r-3} \right] = k \cdot 2^{k-2}.$$

□

Remark 1.4.16. An easy combinatorial computation shows that the set $\overline{\mathscr{G}}$ in the proof of Theorem 1.4.15 has cardinality $k \cdot 2^{k-1}$, i.e. that there there are exactly $k \cdot 2^{k-1}$ isomorphisms classes of toric G-constellations. As a consequence, in order to list all the G-constellations, which are $k \cdot 2^{k-1}$, it is enough to look at the $k \cdot 2^{k-2}$ simple chambers instead of all the $k!$ chambers.

We conclude this section with two examples which help us understand the notions just introduced.

Example 1.4.17. In this example we treat the case $G \cong \mathbb{Z}/5\mathbb{Z}$.

The following picture contains a list of the possible shapes of the abstract chamber stairs of simple chambers and, in each case, the shapes of the G-stairs associated to the toric G-constellations belonging to the respective simple chamber.

[1]In a more suggestive way, this procedure might be called "combinations with repetition of $2r-2$ elements of class $k-2r+1-j$".

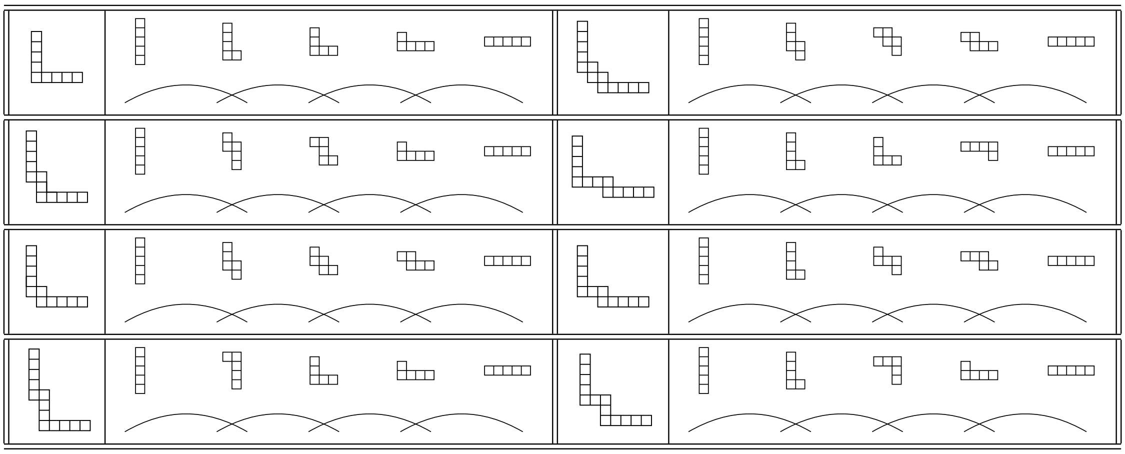


Figure 1.20. Description of the simple chambers for the action of $\mathbb{Z}/5\mathbb{Z}$.

As predicted by Theorem 1.4.15, the possible shapes for the chamber stairs of simple chambers are $8 = 2^{5-2}$, and there are 5 different ways to label each chamber stair.

Example 1.4.18. In this example we treat the case $G \cong \mathbb{Z}/4\mathbb{Z}$.

The following picture contains a list of the possible shapes of the abstract chamber stairs and, in each case, the shapes of the G-stairs associated to the toric G-constellations belonging to the respective chamber.

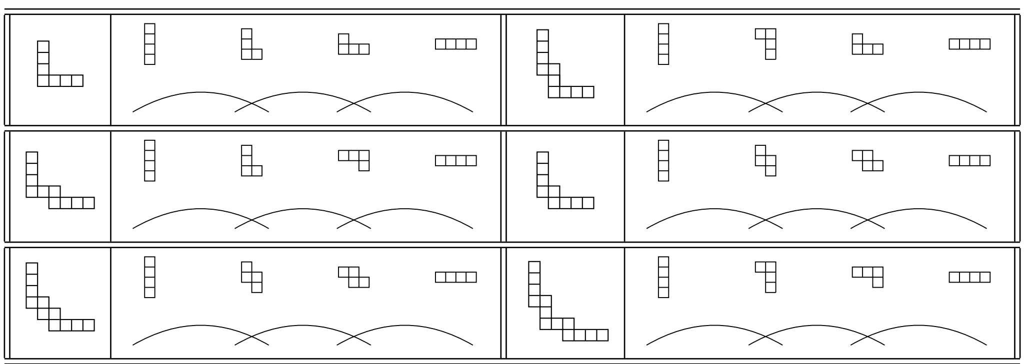


Figure 1.21. Description of the chambers for the action of $\mathbb{Z}/4\mathbb{Z}$.

Notice that the first $4 = 2^{4-2}$ pictures correspond to simple chambers. Moreover, as predicted by Theorem 1.3.17, the possible shapes for the chamber stairs are $6 = (4-1)!$, and there are 4 different ways to label each chamber stair.

Note also that, after having labeled each box appropriately, the first and last chambers in Figure 1.21 correspond to C_G and C_G^{OP} respectively (see Example 1.4.14).

1.5 The costruction of the tautological bundles $\mathscr{R}_C$

The quasi projective variety $\mathscr{M}_C$ is a fine moduli space obtained by GIT as described in [45] by King. In particular, there exists a universal family $\mathscr{U}_C \in \mathrm{Ob}\,\mathrm{Coh}(\mathscr{M}_C \times \mathbb{A}^2)$. The tautological bundle is the pushforward

$$\mathscr{R}_C = (\pi_{\mathscr{M}_C})_*(\mathscr{U}_C).$$

Recall that (see Remark 1.0.20) it is a vector bundle of rank $k = |G|$ whose fibres are G-constellations and, more precisely, over each point $[\mathscr{F}] \in \mathscr{M}_C$ the fibre $(\mathscr{R}_C)_{[\mathscr{F}]}$ is canonically isomorphic to the space of global sections $H^0(\mathbb{A}^2, \mathscr{F})$.

In this section we will give an explicit construction of the tautological bundles $\mathscr{R}_C$ for all chambers $C \subset \Theta^{\mathrm{gen}}$ in terms of their chamber stairs. We will adopt the same notation as in Section 1.3.1.

The following proposition is the key result that we will use in this section.

Proposition 1.5.1 ([29, Proposition 2.4.]). *Let $\pi : \mathbb{A}^2 \to X$ be the projection map where $X = \mathbb{A}^2/G$ and $G \subset \mathrm{Sl}(2,\mathbb{C})$ is any (possibly nonabelian) finite subgroup.*

Let $\varepsilon : Y \to X$ be the crepant resolution of singularities of X. We denote by $\mathscr{O}' = \mathscr{O}_{\mathbb{A}^2} \underset{\mathscr{O}_X}{\otimes} \mathscr{O}_Y = \varepsilon^*\pi_*\mathscr{O}_{\mathbb{A}^2}$ and by $\widetilde{\mathscr{O}} = \mathscr{O}'/\mathrm{Tor}_{\mathscr{O}_Y}\mathscr{O}'$. Then, the $\mathscr{O}_Y$-module $\widetilde{\mathscr{O}}$ is locally free of rank $|G|$.

After some preliminary results, we shall prove the following generalisation of Proposition 1.5.1.

Theorem 1.5.2. *Let $\pi : \mathbb{A}^2 \to X$ be the projection map where $X = \mathbb{A}^2/G$ and $G \subset \mathrm{Sl}(2,\mathbb{C})$ is an abelian finite subgroup.*

Let $\varepsilon : Y \to X$ be the crepant resolution of singularities of $\mathbb{A}^2/G$ and let $\mathscr{K} \subset \mathscr{O}_{\mathbb{A}^2}$ be a coherent (G-invariant) monomial ideal sheaf. We denote by $\mathscr{K}'$ the sheaf $\mathscr{K}' = \mathscr{K} \underset{\mathscr{O}_X}{\otimes} \mathscr{O}_Y \cong \varepsilon^*\pi_*\mathscr{K}$ and we consider $\widetilde{\mathscr{K}} = \mathscr{K}'/\mathrm{Tor}_{\mathscr{O}_Y}\mathscr{K}'$. Then, the $\mathscr{O}_Y$-module $\widetilde{\mathscr{K}}$ is locally free of rank $|G|$.

Notation 1.6. From now on, given a coherent monomial ideal sheaf $\mathscr{K} \subset \mathscr{O}_{\mathbb{A}^2}$, we will denote by $\widetilde{\mathscr{K}}$ the $\mathscr{O}_Y$-module defined by

$$\widetilde{\mathscr{K}} = \varepsilon^*\pi_*\mathscr{K}/\mathrm{Tor}_{\mathscr{O}_Y}\varepsilon^*\pi_*\mathscr{K}.$$

Lemma 1.6.1. *Suppose that $\mathscr{K}$ is generated by the monomials $x^{\alpha_1}y^{\beta_1},\dots,x^{\alpha_s}y^{\beta_s}$. Then, on each toric chart $U_j \subset Y$ with coordinates (a_j, c_j), the sheaf $\widetilde{\mathscr{K}}$ agrees with the sheaf $\mathscr{H}_j$ associated to the $\mathbb{C}[a_j,c_j]$-module:*

$$H_j = H^0(U_j,\mathscr{R}_C) = \frac{K_j}{K_j \cap I_j} \subset \frac{\mathbb{C}[a_j,c_j,x,y]}{K_j \cap I_j},$$

where K_j and I_j are the ideals of $\mathbb{C}[a_j,c_j,x,y]$ given by

$$K_j = (x^{\alpha_1}y^{\beta_1},\dots,x^{\alpha_s}y^{\beta_s})$$

and

$$I_j = (a_j y^{k-j} - x^j, c_j x^{j-1} - y^{k-j+1}, a_j c_j - xy),$$

and the gluings on the intersections $U_i \cap U_j$, for $1 \le i, j \le k$, are given by:

$$\begin{array}{ccc}
\Gamma(U_i \cap U_j, \mathscr{H}_i) & \xrightarrow{\varphi_{ij}} & \Gamma(U_i \cap U_j, \mathscr{H}_j) \\
x & \longmapsto & x, \\
y & \longmapsto & y, \\
a_i & \longmapsto & a_j^{i-j+1} c_j^{i-j}, \\
c_i & \longmapsto & a_j^{j-i} c_j^{j-i+1}.
\end{array}$$

Proof. The proof is achieved by direct computation, after noticing that the gluings on the intersections are deduced from the toric description of the toric quasiprojective variety $\mathscr{M}_C$ given at the beginning of Section 1.3.1 and, in particular, from Equations 1.1.4. □

Remark 1.6.2. If $x^{\alpha_1} y^{\beta_1}, \dots, x^{\alpha_s} y^{\beta_s}$ are the generators of some C-stair Γ_C and $\mathscr{K}$ is defined as in Lemma 1.6.1, all the G-sFd associated to the toric fibres of $\widetilde{\mathscr{K}}$ are substairs of Γ_C. This is a consequence of Nakayama's Lemma together with the following three facts:

$$\begin{array}{ll}
\forall j = 1, \dots, k, \forall i = 1, \dots, s & x^{\alpha_i+1} y^{\beta_i+1} \in (K_j \cap I_j) + (a_j, c_j), \\
\forall j = 1, \dots, k, & x^{\alpha_1} y^{\beta_1+k} \in (K_j \cap I_j) + (a_j, c_j), \\
\forall j = 1, \dots, k, & x^{\alpha_s+k} y^{\beta_s} \in (K_j \cap I_j) + (a_j, c_j).
\end{array} \tag{1.6.1}$$

The relations 1.6.1 follow from the easy observations that

$$x^{\alpha_i} y^{\beta_i} \cdot (a_j c_j - xy) = a_j c_j x^{\alpha_i} y^{\beta_i} - x^{\alpha_i+1} y^{\beta_i+1} \in K_j \cap I_j,$$
$$y^{j-1} \cdot x^{\alpha_1} y^{\beta_1} \cdot (c_j x^{j-1} - y^{k-j+1}) = c_j x^{\alpha_1+j-1} y^{\beta_1+j-1} - x^{\alpha_1} y^{\beta_1+k} \in K_j \cap I_j,$$
$$x^{k-j} \cdot x^{\alpha_s} y^{\beta_s} \cdot (a_j y^{k-j} - x^j) = a_j x^{\alpha_s+k-j} y^{\beta_s+k-j} - x^{\alpha_s+k} y^{\beta_s} \in K_j \cap I_j.$$

We are now in position to prove Theorem 1.5.2.

Proof. (*of Theorem 1.5.2*). By construction, $\widetilde{\mathscr{K}}$ agrees with $\mathscr{O}_Y^{\oplus|G|}$ outsides the exceptional divisor. Moreover, since by definition $\widetilde{\mathscr{K}}$ is torsion free, by [24, §2 Prop. 20], we have an injective morphism of sheaves

$$\widetilde{\mathscr{K}} \xhookrightarrow{\psi} \mathscr{E}$$

for some locally free sheaf $\mathscr{E}$ of rank k. We want to show that, at the stalks level, the morphism ψ induces inclusions

$$\widetilde{\mathscr{K}}_p \xhookrightarrow{\psi_p} \bigoplus_{i=1}^{|G|} \mathscr{O}_{Y,p}$$

for all $p \in Y$, such that the image of each ψ_p is a direct sum of principal, eventually non-proper, ideals of $\mathscr{O}_{Y,p}$.

Let us restrict to the toric chart U_j for some $j = 1, \dots, k$. We will adopt the notation of Lemma 1.6.1. It is enough to study the relations among the generators of the stalk over the origin of U_j, because the locus where the sheaf $\widetilde{\mathscr{K}}$ fails to be locally free must be a toric subvariety of Y. Therefore, we focus on the $\mathbb{C}[a_j,c_j]_{(a_j,c_j)}$-module $\mathscr{H}_{j\,0_j}$ where $0_j \in U_j$ is the origin and $\mathscr{H}_j$ is the sheaf associated to the $\mathbb{C}[a_j,c_j]$-module H_j defined in Lemma 1.6.1. Let $\{m_1,\dots,m_N\}$ be a minimal set of generators of $\mathscr{H}_{j\,0_j}$ made of monomials in the variables x and y. Notice that the generators of the ideal $K_j \cap I_j$ have one of the following forms

$$\begin{gathered} a_j x^\alpha y^{\beta+k-j} - x^{\alpha+j} y^\beta, \\ c_j x^{\alpha+j-1} y^\beta - x^\alpha y^{\beta+k-j+1}, \\ a_j c_j x^\alpha y^\beta - x^{\alpha+1} y^{\beta+1}, \end{gathered} \tag{1.6.2}$$

for some $\alpha,\beta \in \mathbb{N}$ such that the two monomials in the variables x and y appearing in each binomial belong to K_j. To conclude, it is enough to prove that there are no relations with coefficients in $\mathbb{C}[a_j,c_j]$ between the generators m_i for $i = 1,\dots,N$.

First recall that, if a monomial $x^\alpha y^\beta$ belongs to K_j, then the monomial $x^{\alpha+1} y^{\beta+1}$ is in $K_j \cap I_j + (a_j,c_j)$ (see Remark 1.6.2).

Now we show that there are no degree one relations between the m_i's. Indeed, a degree one relation with coefficients in $\mathbb{C}[a_j,c_j]$ must be of the following form

$$c_j x^\alpha y^\beta - a_j x^{\alpha-2j+1} y^{\beta+2k-2j+1}. \tag{1.6.3}$$

Suppose by contradiction that there exist i and j such that

$$m_i = x^\alpha y^\beta \text{ and } m_j = x^{\alpha-2j+1} y^{\beta+2k-2j+1}.$$

By manipulating appropriately the generators in (1.6.2), we also obtain

$$x^\alpha y^\beta - a_j^2 x^{\alpha-2j} y^{\beta+2k-2j}$$

which, together with (1.6.3), implies that

$$a_j(a_j c_j - xy) x^{\alpha-2j} y^{\beta+2k-2j} \in K_j \cap I_j.$$

This tells us that $x^{\alpha-2j} y^{\beta+2k-2j} \in K_j$ and, by Remark 1.6.2, that

$$x^{\alpha-2j+1} y^{\beta+2k-2j+1} \in K_j \cap I_j + (a_j,c_j)$$

i.e., by Nakayama's Lemma, that $x^{\alpha-2j+1} y^{\beta+2k-2j+1}$ does not belong to the minimal set of generators $\{\, m_i \mid i = 1,\dots,N \,\}$. Similarly one proves that there are no higher degree relations between the m_i's and, as a consequence, that $N = k$. □

Remark 1.6.3. As expected in dimension 3, Theorem 1.5.2 is, in general, false. For instance, given the $(\mathbb{Z}/2\mathbb{Z})^2$ action over $\mathbb{A}^3$ defined by the inclusion

$$\begin{aligned} (\mathbb{Z}/2\mathbb{Z})^2 &\longrightarrow \mathrm{Sl}(3,\mathbb{C}) \\ (1,0) &\longmapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \\ (0,1) &\longmapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \end{aligned}$$

the quotient singularity $X = \mathbb{A}^3/(\mathbb{Z}/2\mathbb{Z})^2$ admits four different crepant resolutions $\varepsilon_i : Y_i \to X$, for $i = 1,\dots,4$. All of them are toric and they are described by the planar graphs in Figure 1.22. These diagrams are obtained by considering a fan Σ_i for each resolution Y_i then, each simplex

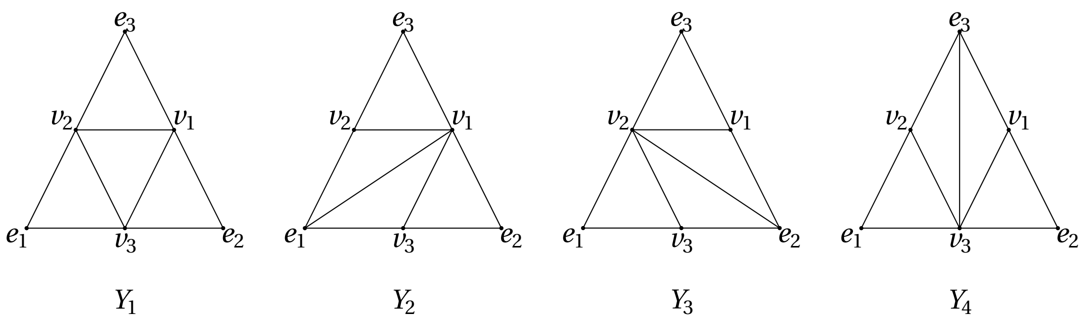


Figure 1.22. Toric description of the crepant resolutions of $\mathbb{A}^3/(\mathbb{Z}/2\mathbb{Z})^2$.

in the planar graph is the intersection of a cone in Σ_i, with the plane containing the heads of the rays that generate Σ_i. Notice that Y_1 differs from the other resolutions by just one flop $Y_1 \overset{\sigma_i}{\dashleftarrow\dashrightarrow} Y_i$ for $i = 2,3,4$.

Now, let $\widetilde{\mathscr{O}}_i$, for $i = 1,\dots,4$, be the torsion free $\mathscr{O}_{Y_i}$-module defined by

$$\widetilde{\mathscr{O}}_i = \varepsilon_i^* \pi_* \mathscr{O}_{\mathbb{A}^3} / \mathrm{Tor}_{\mathscr{O}_{Y_i}} \varepsilon_i^* \pi_* \mathscr{O}_{\mathbb{A}^3},$$

where $\pi : \mathbb{A}^3 \to X$ is the canonical projection. A direct computation shows that only $\widetilde{\mathscr{O}}_1$ is locally free, and, for $i = 2,3,4$, the locus where $\widetilde{\mathscr{O}}_i$ fails to be locally free coincides with the line flopped by σ_i. In this setting, it can be shown that the pair $(Y_1, \widetilde{\mathscr{O}}_i)$ is canonically isomorphic to the pair $((\mathbb{Z}/2\mathbb{Z})^2-\mathrm{Hilb}(\mathbb{A}^3), \mathscr{R})$ where $\mathscr{R}$ is the tautological bundle.

In this last part of Section 1.5 we state and prove the last main theorem. Before to give the proof, we will also state and prove some corollaries and results needed in the proof.

Theorem 1.6.4. *Let $C \subset \Theta^{\mathrm{gen}}$ be a chamber and let $\Gamma_C \subset \mathscr{T}_G$ be a C-stair. Suppose that Γ_C has $s \geq 1$ ordered (see Remark 1.2.20) generators $v_1,\dots,v_s$ with associated monomials*

$$x^{\alpha_1} y^{\beta_1}, \dots, x^{\alpha_s} y^{\beta_s} \in \mathbb{C}[x,y].$$

Consider the ideal sheaf $\mathscr{K} = (x^{\alpha_1} y^{\beta_1}, \dots, x^{\alpha_s} y^{\beta_s})\mathscr{O}_{\mathbb{A}^2}$, then

$$\mathscr{R}_C \cong \varepsilon^* \pi_* \mathscr{K} / \mathrm{Tor}_{\mathscr{O}_{\mathscr{M}_C}}(\varepsilon^* \pi_* \mathscr{K}).$$

The following corollary is a direct consequence of Theorem 1.6.4 Lemma 1.6.1

Corollary 1.6.5. *On each toric chart $U_j \subset \mathscr{M}_C$ with coordinates (a_j, c_j), the tautological bundle $\mathscr{R}_{C|_{U_j}}$ agrees with the sheaf $\mathscr{H}_j$ associated to the $\mathbb{C}[a_j, c_j]$-module H_j in Lemma 1.6.1.*

Remark 1.6.6. For the trivial ideal $K = (1) = \mathbb{C}[x,y]$ Corollary 1.6.5 recovers Nakamura's description of the G-Hilbert scheme when G is abelian (see [52]).

Remark 1.6.7. Notice that, over the origin of the first and the last charts, the $\mathscr{O}_{U_1}$-module $\mathscr{H}_1$ and the $\mathscr{O}_{U_k}$-module $\mathscr{H}_k$ have, as toric fibres, the expected G-constellations $\mathscr{F}_1$ and $\mathscr{F}_k$, i.e

$$\mathscr{F}_1 \cong \mathscr{H}_{1 0_1} \cong \frac{(x^{\alpha_1} y^{\beta_1})}{(x^{\alpha_1} y^{\beta_1+k}, x^{\alpha_1+1} y^{\beta_1})} \subset \frac{\mathbb{C}[x,y]}{(x^{\alpha_1} y^{\beta_1+k}, x^{\alpha_1+1} y^{\beta_1})}$$

and

$$\mathscr{F}_k \cong \mathscr{H}_{k 0_k} \cong \frac{(x^{\alpha_s} y^{\beta_s})}{(x^{\alpha_s+k} y^{\beta_s}, x^{\alpha_s} y^{\beta_s+1})} \subset \frac{\mathbb{C}[x,y]}{(x^{\alpha_s+k} y^{\beta_s}, x^{\alpha_s} y^{\beta_s+1})},$$

where $0_i \in U_i$ is, for $i = 1, k$, the origin.

We prove this only for the origin of U_k, the other proof is similar. We start by showing that

$$x^{\alpha_i} y^{\beta_i} \in (K_k \cap I_k) + (a_k, c_k) \text{ for } i = 1, \dots, s-1.$$

Notice that, for all $i = 1, \dots, s-1$, we have

$$\alpha_i \geq 0, \quad \beta_i > \beta_{i+1} > \beta_s \geq 0, \quad \alpha_i + k - 1 \geq \alpha_{i+1}.$$

Therefore, we can write:

$$c_k x^{\alpha_i+k-1} y^{\beta_i-1} - x^{\beta_i} y^{\alpha_i} = \begin{cases} c_k x^{\alpha_i+k-1-\alpha_{i+1}} y^{\beta_i-1-\beta_{i+1}} (x^{\alpha_{i+1}} y^{\beta_{i+1}}) - x^{\alpha_i} y^{\beta_i} \\ (x^{\alpha_i} y^{\beta_i-1})(c_k x^{k-1} - y) \end{cases}$$

which implies

$$x^{\alpha_i} y^{\beta_i} \in (K_k \cap I_k) + (a_k, c_k)\ \forall i = 1, \dots, s-1.$$

Now, we have

$$K_k \cap I_k + (a_k, c_k) = (x^{\alpha_s} y^{\beta_s}) \cap I_k + (a_k, c_k) = (x^{\alpha_s} y^{\beta_s}) \cdot I_k + (a_k, c_k) = (x^{\alpha_s+k} y^{\beta_s}, x^{\alpha_s} y^{\beta_s+1}, a_k, c_k),$$

which gives

$$\mathscr{H}_{k 0_k} \cong \frac{(x^{\alpha_s} y^{\beta_s})}{(x^{\alpha_s+k} y^{\beta_s}, x^{\alpha_s} y^{\beta_s+1}, a_k, c_k)} \subset \frac{\mathbb{C}[x,y,a_k,c_k]}{(x^{\alpha_s+k} y^{\beta_s}, x^{\alpha_s} y^{\beta_s+1}, a_k, c_k)}.$$

Corollary 1.6.8. *Let C and $\mathscr{K}$ be as in Theorem 1.6.4. Then, $\mathscr{M}_C$ can be identified with a closed G-invariant subvariety of* $\mathrm{Quot}^{|G|}_{\mathscr{K}}(\mathbb{A}^2)$.

Definition 1.6.9. Let $K \subset \mathbb{C}[x,y]$ be the ideal generated by the (ordered) set of monomials

$$\left\{ x^{\alpha_i} y^{\beta_i} \;\middle|\; i = 1, \dots, s \right\}$$

associated to the generators of some chamber stair Γ_C and let $\Gamma_K = \{(m,i) \in \mathscr{T}_G \mid m \in K\}$ be the subset of the representation tableau corresponding to K. Given a monomial $m_b \in K$ corresponding to a box $b \in \Gamma_C \subset \Gamma_K$, we will say that:

- the property **(A**$_j$**)** holds for m_b (or for b) if

$$x^{-j} y^{k-j} \cdot m_b \in \Gamma_K,$$

- the property $(\mathbf{C}_j)$ holds for m_b (or for b) if

$$x^{j-1}y^{-k+j-1}\cdot m_b \in \Gamma_K.$$

Example 1.6.10. Consider the action of $\mathbb{Z}/4\mathbb{Z}$ on $\mathbb{A}^2$ induced by the representation (1.1.2). Let $\theta \in \Theta^{\mathrm{gen}}$ be the generic stability condition $\theta = (-2,1,2,-1)$ (see Remark 1.3.18) and C the chamber containing it.Then, a chamber stair $\Gamma_C \subset \mathscr{T}_G$ for C is depicted in Figure 1.23. Moreover, the number j appears in red (resp. in blue) in a box if the property $(\mathbf{A}_j)$ (resp. $(\mathbf{C}_j)$) holds for that box.

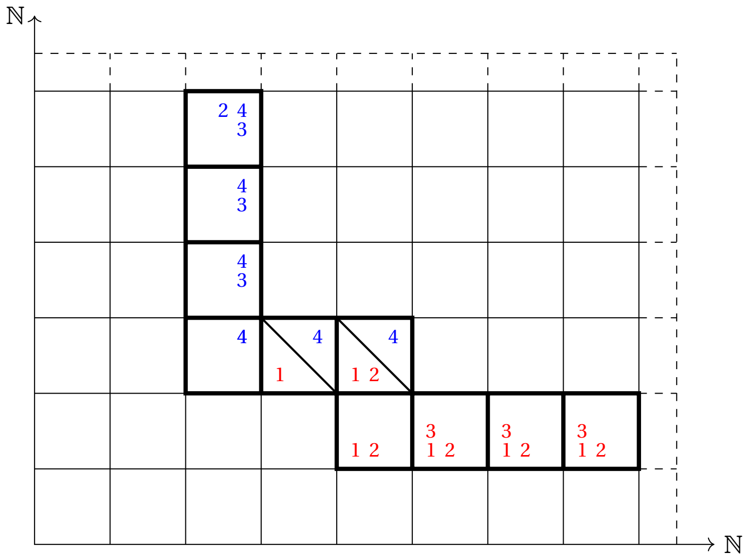


Figure 1.23. The property $(\mathbf{A}_j)$ (resp. $(\mathbf{C}_j)$) holds for a box, if the number j appears in red (resp. blue) in the box.

The abstract G-stairs of the toric C-stable G-constellations are listed in Figure 1.24. In particular, C is a simple chamber and the third toric G-constellation is C-characteristic.

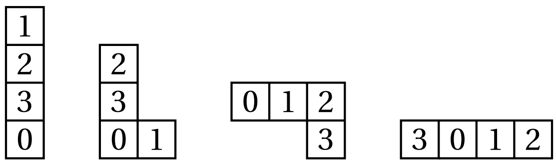


Figure 1.24. The abstract C-stairs of the toric C-stable G-constellations.

Lemma 1.6.11. *If the property* $(\mathbf{A}_j)$ *(resp.* $(\mathbf{C}_j)$*) holds for a box* $b \in \Gamma_C$ *then it holds also for the box after (resp. before)* b.

Proof. Let $m_b = x^\alpha y^\beta$ be the monomial associated to the box b. From Definition 1.6.9, it follows immediately that, if the property $(\mathbf{A}_j)$ (resp. $(\mathbf{C}_j)$) holds for b, then it holds for all the monomials $x^\gamma y^\delta$ such that $\gamma \geq \alpha$ and $\delta \geq \beta$. This proves the Lemma in the case in which the box after (resp. before) b is on the right (resp. above) b.

We prove the remaining case for the property $(\mathbf{C}_j)$ and we leave the similar proof for $(\mathbf{A}_j)$. We have to prove that, if two monomials of the form $x^\alpha y^\beta, x^{\alpha-1}y^\beta$ correspond to some successive boxes in Γ_C and the property $(\mathbf{C}_j)$ holds for $x^\alpha y^\beta$ then it holds also for $x^{\alpha-1}y^\beta$. In other words, we suppose that

$$m_1 = x^{\alpha+j-1}y^{\beta-k+j-1} \in K,$$

and we want to prove that

$$m_2 = x^{\alpha+j-2}y^{\beta-k+j-1} \in K.$$

Let b_1, b_2 be the boxes corresponding to m_1, m_2 and let b be the box corresponding to $x^{\alpha-1}y^\beta$. If $b_1 \in \Gamma_K \smallsetminus \Gamma_C$ it follows easily that $b_2 \in \Gamma_K$. Suppose $b_1 \in \Gamma_C$ and consider the connected substair $\Gamma \subset \Gamma_C$ whose first box is b and whose last box is b_1. We have, by construction,

$$\mathfrak{w}(\Gamma) = j \text{ and } \mathfrak{h}(\Gamma) = k - j + 2,$$

which imply that Γ contains $k+1$ boxes.

Let $\Gamma' = \Gamma \smallsetminus \{b_1\}$ be the connected G-substair of Γ_C obtained by removing the last box from Γ and let $b' \in \Gamma_C$ be the last box of Γ'. Now, by construction, b is a vertical left cut for Γ' in Γ_C and, as a consequence of Remark 1.4.5 also b' is a vertical cut. Therefore b' must correspond to the monomial m_2 from which it directly follows

$$b' = b_2 \in \Gamma_C,$$

which implies the thesis. □

Proof. (*of Theorem 1.6.4*). Let $\widetilde{\mathscr{K}}$ be the sheaf defined in the statement, i.e.

$$\widetilde{\mathscr{K}} \cong \varepsilon^*\pi_*\mathscr{K} / \mathrm{Tor}_{\mathscr{O}_{\mathscr{M}_C}}(\varepsilon^*\pi_*\mathscr{K}).$$

Theorem 1.5.2 implies that $\widetilde{\mathscr{K}}$ is locally free, i.e. it is a vector bundle. Moreover, if we endow the product $\mathscr{M}_C \times \mathbb{A}^2$ with the G-action defined by

$$\begin{array}{ccc} G \times \mathscr{M}_C \times \mathbb{A}^2 & \longrightarrow & \mathscr{M}_C \times \mathbb{A}^2 \\ (g_k^i, p, (x,y)) & \longmapsto & (p, (\xi_k^{-i}x, \xi_k^i y)). \end{array}$$

where g_k is the (fixed) generator of the cyclic group G (see subsection 1.1.1), it turns out that the $\mathscr{O}_{\mathscr{M}_C \times \mathbb{A}^2}$-module $\widetilde{\mathscr{K}}$ is G-equivariant with respect to this action. The last observation, together with Remark 1.6.7, implies that

$$\widetilde{\mathscr{K}} \cong \mathscr{O}_{\mathscr{M}_C}[G],$$

whose proof, at this point, is identical to the proof of [39, Lemma 9.4].

To prove the theorem, we will use the description of $\widetilde{\mathscr{K}}$ given in Corollary 1.6.5.

We know from Remark 1.0.13 that the tautological bundles $\mathscr{R}_C$ and $\mathscr{R}_{C_G}$ agree on the complement U_C of the exceptional locus of $\mathscr{M}_C$. Moreover, we have, as a consequence of the construction of $\widetilde{\mathscr{K}}$ and of Remark 1.6.6, isomorphisms

$$\mathscr{R}_{C|_{U_C}} \cong \mathscr{R}_{C_G|_{U_C}} \cong \widetilde{\mathscr{K}}_{|_{U_C}} \cong \mathscr{O}_{U_C}^{\oplus k}.$$

Now we show that the fibres of $\mathscr{R}_C$ and $\widetilde{\mathscr{K}}$ over the toric points of $\mathscr{M}_C$ are the same G-constellations. This will be enough to prove the statement, because each chamber is uniquely identified by its toric G-constellations. We split this part in several steps:

STEP 0 Over each point of $p \in \mathscr{M}_C$ the fibre $\widetilde{\mathscr{K}}_p$ is a G-equivariant $\mathbb{C}[x,y]$-module and, over each origin $0_j \in U_j$ the fibre $\widetilde{\mathscr{K}}_{0_j}$ is also $\mathbb{T}^2$-equivariant. This follows from the fact that the ideal K_j is generated by monomials and that the ideal I_j is generated by G-eigenbinomials (recall that the group G acts trivially on U_j) of positive degrees in the variables a_j, c_j.

STEP 1 All the G-sFd associated to the toric fibres of $\widetilde{\mathscr{K}}$ are substairs of the C-stair Γ_C. For this, see Remark 1.6.2.

STEP 2 For all $j = 1,\dots,k$, the j-th toric G-constellation $\widetilde{\mathscr{K}}_{0_j}$ is irreducible. Let $\Gamma_j \subset \Gamma_C$ be the G-sFd associated to $\widetilde{\mathscr{K}}_{0_j}$. Then, the G-constellation $\widetilde{\mathscr{K}}_{0_j}$ is irreducible if and only if Γ_j is connected.

First observe that, for a box $b \in \Gamma_C$ both the properties $(\mathbf{A}_j)$ and $(\mathbf{C}_j)$ implies that the corresponding monomial m_b belongs to $(K_j \cap I_j) + (a_j, c_j)$. This is true because, if $m_b = x^\alpha y^\beta$, then

$$\begin{aligned} (\mathbf{A}_j) \Rightarrow \quad & a_j x^{\alpha-j} y^{\beta+k-j} - x^\alpha y^\beta \in K_j \cap I_j, \\ (\mathbf{C}_j) \Rightarrow \quad & c_j x^{\alpha+j-1} y^{\beta-k+j-1} - x^\alpha y^\beta \in K_j \cap I_j. \end{aligned} \tag{1.6.4}$$

On the other hand, $b \in \Gamma_C \smallsetminus \Gamma_j$ if and only if $m_b \in (K_j \cap I_j) + (a_j, c_j)$. In particular, by construction, at least one of the following relations is true.

1. $a_j x^{\alpha-j} y^{\beta+k-j} - x^\alpha y^\beta \in K_j \cap I_j$,
2. $c_j x^{\alpha+j-1} y^{\beta-k+j-1} - x^\alpha y^\beta \in K_j \cap I_j$,
3. $a_j c_j x^{\alpha-1} y^{\beta-1} - x^\alpha y^\beta \in K_j \cap I_j$.

Notice that $b \in \Gamma_C$ implies (see STEP 1) that (3) cannot hold true. Therefore, given $b \in \Gamma_C$, it belongs to Γ_j if and only if one among the two properties $(\mathbf{A}_j)$ and $(\mathbf{C}_j)$ holds for b. Now, the connectedness of Γ_j is a consequence of Lemma 1.6.11.

STEP 3 For all $j = 1,\dots,k$, the j-th toric G-constellation $\widetilde{\mathscr{K}}_{0_j}$ has width $\mathfrak{w}(\widetilde{\mathscr{K}}_{0_j}) = j$. Let $\Gamma_j \subset \Gamma_C$ be, as in the previous step, the G-sFd associated to $\widetilde{\mathscr{K}}_{0_j}$, and let $x^\alpha y^\beta, x^\gamma y^\delta$ be the monomials in $\mathbb{C}[x,y] \subset \mathbb{C}[a_j, c_j, x, y]$ corresponding to the first and the last box of Γ_j. Suppose that, for some $\beta + j - k \le \beta' \le \beta$ and $\gamma - j + 1 \le \gamma' \le \gamma$ we have

$$x^{\alpha+j} y^{\beta'}, x^{\gamma'} y^{\delta+k-j+1} \in K_j.$$

Then, the following relations

$$\begin{aligned} a_j x^\alpha y^{\beta'+k-j} - x^{\alpha+j} y^{\beta'} \in K_j \cap I_j, \\ c_j x^{\gamma'+j-1} y^\delta - x^{\gamma'} y^{\delta+k-j+1} \in K_j \cap I_j, \end{aligned}$$

imply that

$$\begin{aligned} x^{\alpha+j} y^{\beta'} \in K_j \cap I_j + (a_j, c_j), \\ x^{\gamma'} y^{\delta+k-j+1} \in K_j \cap I_j + (a_j, c_j). \end{aligned} \tag{1.6.5}$$

As a consequence of the relations 1.6.5, we have

$$\mathfrak{w}(\widetilde{\mathscr{K}}_{0_j}) \le j,$$

and,

$$\mathfrak{h}(\widetilde{\mathscr{K}}_{0_j}) \leq k - j + 1.$$

The equality $\mathfrak{w}(\widetilde{\mathscr{K}}_{0_j}) = j$ follows from the fact that, as per the previous steps, $\widetilde{\mathscr{K}}_{0_j}$ is an irreducible toric G-constellation, hence

$$\mathfrak{w}(\widetilde{\mathscr{K}}_{0_j}) + \mathfrak{h}(\widetilde{\mathscr{K}}_{0_j}) = k + 1.$$

STEP 4 As an immediate consequence of the previous step all the G-constellations $\widetilde{\mathscr{K}}_{0_j}$, for $j = 1, \ldots, k$, are different to each other.

Now, the above listed properties imply that $\widetilde{\mathscr{K}}$ is the tautological bundle $\mathscr{R}_{C'}$ of some chamber $C' \subset \Theta^{\text{gen}}$ which admits Γ_C as C'-stair and this, by Lemma 1.4.6, implies $C' = C$. □

1.7 A recent proof of the conjecture

Very recently, a proof of the Craw–Ishii conjecture (see Conjecture 1.0.24) appeared in two preprints [70, 71]. In particular, in [70], the abelian case is revisited and treated in dimension higher than 3 and, in [70], the conjecture is proven in a non-abelian setting. Unfortunately, there has been no time to take account of these results in this thesis. We use this section to comment a little on these preprints and to relate them to [30].

The main result in [70] is the following.

Theorem 1.7.1. *([70, Theorem 5.2]) Let $Y \to X$ be a crepant resolution of a quotient singularity $X = \mathbb{A}^n/G$, for $G < \mathrm{Sl}(n, \mathbb{C})$ abelian and finite. Then, there exists a generic stability condition $\theta \in \Theta^{\text{gen}}$ such that the normalisation of $\mathscr{M}_\theta$ and Y are isomorphic over X if and only if X admits a family of G constellations which, outside the exceptional locus, agrees with the tautological family over X and, over the exceptional divisor, has only irreducible G-constellations.*

This result confirms the important role of irreducible G-constellations, as we mentioned in Remark 1.0.16. In other words, Theorem 1.7.1 says that certain families of irreducible G-constellations on X induce canonical isomorphisms with (the normalisation) of some moduli spaces. This approach is very similar to the one we tried to prove Craw–Ishii's conjecture. Indeed, our idea was precisely to generalize Theorem 1.5.2 to the 3-dimensional and non-abelian case in order to build families of (irreducible) G-constellations on the crepant resolutions of X which, outside the exceptional locus, agrees with the tautological family over X.

The proof of Craw–Ishii's conjecture is in [71].

Theorem 1.7.2 ([71, Theorem 4.1])**.** *Let $G \subset \mathrm{Sl}(3, \mathbb{C})$ be a finite subgroup and let $Y \to \mathbb{A}^3/G$ be a crepant resolution. Then, there is a generic stability condition $\theta \in \Theta^{\text{gen}}$ such that $Y \cong \mathscr{M}_\theta$.*

The technology used in the proof in Theorem 1.7.1 is different from that used in [71], and it is more similar to that used in Craw–Ishii's paper [15], where Conjecture 1.0.24 was stated.

It would be interesting in the future to combine the new ideas in [70, 71] and those in [30] to work in higher dimensions. In that case, it is not true in general that there exists a crepant resolution and, even in that case, the moduli spaces $\mathscr{M}_\theta$, for $\theta \in \Theta^{\text{gen}}$, may be singular. Therefore, it is not possible to generalize Theorem 1.7.2 as it is stated. Using our approach, however, some generalisations may be obtained. Indeed, a generalisation of Theorem 1.5.2 without the hypothesis of abelianity and without limitations on the dimension, would allow us to build families of G-constellations with the properties in Theorem 1.7.1 on certain terminalisations Y of the singularity X.

We hope that these families may also give isomorphisms with (the normalisation) of some moduli spaces $\mathscr{M}_\theta$ for $\theta \in \Theta \in^{\text{gen}}$.

1.8 A crepant resolution for $G = H_{168} < \mathrm{Sl}(3,\mathbb{C})$

As already mentioned in the introduction, the finite subgroups of $\mathrm{Sl}(3,\mathbb{C})$ are listed (see [72]) and they consists of some infinite families and some sporadic groups. Among the latter appears the *Klein group* $H_{168} \cong \mathbb{P}\mathrm{SL}(2,\mathbb{F}_7)$, denoted by (I) in [72], or, more precisely, a 3-dimensional representation

$$H_{168} \overset{\rho}{\hookrightarrow} \mathrm{Sl}(3,\mathbb{C}).$$

In this section we will construct a crepant resolution of the singularity $\mathbb{A}^3/H_{168}$ alternative to the one constructed by Markushevich in [50]. Then we will compare them and we will describe the series of flops that connect them.

In what follows, with abuse of notation, we will denote by H_{168} the image of the representation ρ.

1.8.1 The group and the singularity

A set of generators of H_{168} can be found, for example in [50, 64, 72]. For instance, one can choose

$$g_1 = \begin{pmatrix} \xi & 0 & 0 \\ 0 & \xi^2 & 0 \\ 0 & 0 & \xi^4 \end{pmatrix}, \quad \xi = \exp\left(\frac{2\pi i}{7}\right);$$

$$g_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix};$$

$$g_3 = -\frac{2}{\sqrt{7}} \begin{pmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{pmatrix}, \quad \begin{aligned} \alpha &= \sin\left(\tfrac{8\pi}{7}\right), \\ \beta &= \sin\left(\tfrac{4\pi}{7}\right), \\ \gamma &= \sin\left(\tfrac{2\pi}{7}\right). \end{aligned}$$

It turns out (see [50], where an exhaustive description of the singularity is given) that the ring of invariants $\mathbb{C}[\mathbb{A}^3]^{H_{168}}$ has 4 generators and, as a consequence, the singularity $X = \mathbb{A}^3/H_{168}$ is a hypersurface singularity.

The equation of X is:

$$y^3+1728z^7+1008yz^4t-88y^2zt^2-60032z^5t^3+1088yz^2t^4+22016z^3t^6-256yt^7-2048zt^9-x^2,$$

it is singular along the curves:

$$C_1 = \{x = 27z^2 - 4t^3 = 3y + 8zt^2 = 0\} \text{ and } C_2 = \{x, z^2 + 4t^3 = y - 72zt^2\}.$$

They have the property that, generically, outside a finite number of *dissident points*, are respectively locally (analytically) isomorphic to a trivial family of A_2 and of A_3 surface singularities.[2] Notice that the curves C_1 and C_2 are isomorphic to plane affine cusps and that they intersect at their singular point which is also a dissident point for both the curves.

[2] In [60], this 3-fold singularities are called *cDV singularities* The word dissident is also taken from there.

1.8.2 First change of coordinates

Before starting to blowup out the 3-fold singularity looking for a crepant resolution, let us act with a change of coordinates that makes comfortable to perform the blowup. We act on $\mathbb{A}^4$ via the following biholomorphism:

$$\begin{array}{ccc} \mathbb{A}^4 & \xrightarrow{\quad\psi\quad} & \mathbb{A}^4 \\ (x,y,z,t) & \longmapsto & (x, y+18zt^2, z, t/2). \end{array}$$

We have thus obtained a new equation [3] for X:

(1.8.1) $40zt^9-272z^3t^6-1568z^5t^3+2yt^7-1728z^7-248yz^2t^4-504yz^4t-32y^2zt^2-y^3+x^2.$

Furthermore, after the change of coordinates, the description of the singularities of X is:

$$C_1=\left\{\, x=t^3-54z^2=56zt^2+3y=0 \,\right\} \text{ and } C_2=\left\{\, y=x=t^3+2z^2=0 \,\right\}.$$

1.8.3 Blowup of X along C_2

We are now in position to perform the first blowup.

It was observed in [60] that the operation of blowing up families of DuVal singularities (see Section 1.1.2) that are trivial outside some dissident points is a crepant operation, i.e. the canonical sheaf of the blowup is the bullback of the canonical sheaf of the singularity. Therefore, we can blowup indifferently C_1 or C_2. Clearly, different choices can, in principle, lead to different crepant resolutions of singularities. Markushevich, in [50], started resolving X by blowing up C_1. Since we want a different resolution than the one in [50], we start by blowing up the curve C_2.

Before we begin, let us denote by g the element $g=t^3+2z^2\in\mathbb{C}[x,y,z,t]$ and let us rewrite the equation of X in the following more convenient way

$$x^2-y^3+40zg^3-32z(4zg+yt)^2+2ytg^2.$$

Let X_1 be the blowup of X with centre C_2, i.e. $X_1=\mathrm{Bl}_{C_2}X$ and let $\varepsilon_1: X_1\to X$ be the blowup map. Since C_2 is a complete intersection, we can blow up it by applying [21, Prop. IV-25]. A priori, the blowup X_1 will be covered by 3 affine charts, but a careful analysis shows that, in fact, two charts are enough to cover the whole X_1.

Below, we give a local description of X_1 on each chart.

Local pictures. We denote the two charts by U_1 and U_2.

- **Description of U_1.** The variety U_1 is the hypersurface of $\mathbb{A}^4$, with coordinates a_1,b_1,c_1,d_1, defined by the equation:

$$a_1^2b_1+g_1(40c_1b_1^3+2b_1^2d_1-1)-32c_1(4c_1b_1+d_1)^2b_1=0,$$

[3] Equation (1.8.1) is actually the equation of ψ^*X, but, with abuse of notation, we omitted the pullback symbol for the sake of readability.

where $g_1 = d_1^3 + 2c_1^2$, and the blowup map $\varepsilon_{1,1} = \varepsilon_1|_{U_1} : U_1 \to X$ is:

$$\begin{array}{ccc} U_1 & \xrightarrow{\varepsilon_{1,1}} & X \\ \begin{pmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{pmatrix} & \longmapsto & \begin{pmatrix} a_1[2g_1 b_1(20c_1 b_1 + d_1) - 32(4c_1 b_1 + d_1)^2 + a_1^2] \\ 2g_1 b_1(20c_1 b_1 + d_1) - 32(4c_1 b_1 + d_1)^2 + a_1^2 \\ c_1 \\ d_1 \end{pmatrix}. \end{array}$$

We can now compute the singularities of U_1 by applying the Jacobian criterion and we find that U_1 is singular along the union of the following three (smooth and rational) disjoint curves

$$\widetilde{C}_{1,1} = \{\, a_1 = 108c_1 b_1^3 + 1 = 18c_1 b_1 + d_1 = 0 \,\},\ \widetilde{C}_{2,1} = \{\, a_1 = 32c_1 b_1^3 - 1 = 4c_1 b_1 + d_1 = 0 \,\},$$

$$C_{3,1} = \{\, a_1 = c_1 = d_1 = 0 \,\}.$$

They are generically trivial families of DuVal singularities. In particular, $\widetilde{C}_{1,1}$ and $C_{3,1}$ are families of A_2 singularities, while $\widetilde{C}_{2,1}$ is a family of A_1 singularities. This is true because C_2 and C_1 are respectively, generically trivial families of A_3 and A_2 singularities, while, after excluding a finite number of points, the singularities of U_1 along $C_{3,1}$ are locally (analytically) of the form $a_1^2 b_1 + 2c_1^2 + d_1^3$ which is a trivial family of A_2 singularities.

The exceptional divisor of $\varepsilon_{1,1}$ is the surface $E_1 \subset U_1$ defined as follows

$$E_1 = \{\, g_1 = a_1^2 - 32c_1(4c_1 b_1 + d_1)^2 = 0 \,\}.$$

It is a non-normal surface singular along the curves $\widetilde{C}_{2,1}$ and $C_{3,1}$.

- **Description of U_2.** The variety U_2 is the hypersurface of $\mathbb{A}^4$, with coordinates a_2, b_2, c_2, d_2, defined by the equation:

$$a_2^2 + g_2(40c_2 + 2b_2 d_2 - b_2^3) - 32c_2(4c_2 + b_2 d_2)^2 = 0,$$

where $g_2 = d_2^3 + 2c_2^2$, and the blowup map $\varepsilon_{1,2} = \varepsilon_1|_{U_2} : U_2 \to X$ is:

$$\begin{array}{ccc} U_2 & \xrightarrow{\varepsilon_{1,2}} & X \\ (a_2, b_2, c_2, d_2) & \longmapsto & (a_2 g_2, b_2 g_2, c_2, d_2). \end{array}$$

The chart U_2 is singular along the (smooth and rational) curves

$$\widetilde{C}_{1,2} = \{\, a_2 = b_2^2 - 6d_2 = b_2^3 + 108c_2 = 0 \,\},\ \widetilde{C}_{2,2} = \{\, a_2 = b_2^2 + 8d_2 = b_2^3 - 32c_2 = 0 \,\},$$

$$C_{3,2} = \{\, a_2 = c_2 = d_2 = 0 \,\}.$$

Notice that they intersect, two by two, at the origin $(0,0,0,0) \in \mathbb{A}^4$ and that they are two families of A_2 singularities ($\widetilde{C}_{2,2}$ and $C_{3,2}$) and one A_1 singularities ($\widetilde{C}_{1,2}$) trivial outside some dissident points. This can be understood, for instance, from the other chart.

Finally, the exceptional locus of $\varepsilon_{1,2}$ is the surface

$$E_2 = \left\{ g_2 = a_2^2 - 32c_2(4c_2 + b_2 d_2)^2 = 0 \right\}.$$

It is a non-normal surface singular along $\widetilde{C}_{2,2} \cup C_{3,2}$.

Remark 1.8.1. Basically we will use the chart without dissident points, where the families of singularities are trivial, to understand what kind of singularity we have. The other chart contains the information on how the singularities interplay and therefore the interesting information on the blowup and on the relationship between the exceptional dividers is all there. Since the blowup of a trivial family of DuVal singularities is well understood, in the next section, we will show only the blowup of the chart U_2, where the families of singularities are nontrivial.

Similarly, in the remaining sections of this chapter, we will only show the blowup of the charts in which nontrivial families of DuVal singularities appear.

1.8.4 Blowup of X_1 along C_3

Let X_2 be the blowup of X_1 along C_3 and let $\varepsilon_2 : X_2 \to X$ and $\iota_2 : X_2 \to X_1$ be the blowup maps. As explained in Remark 1.8.1, in order to understand the crepant resolution is enough to study the open subset of X_2 defined as $\widetilde{U}_2 = \mathrm{Bl}_{C_{3,2}} U_2$. We can apply again [21, Prop. IV-25] and, after a direct computation we discover that $\widetilde{U}_2$ is covered by two affine charts W_1, W_2 that we describe below.

Local pictures. As above, we proceed with a local description of $\widetilde{U}_2 = W_1 \cup W_2$.

- **Description of W_1.** The variety W_1 is the hypersurface of $\mathbb{A}^4$, with coordinates $\alpha_1, \beta_1, \gamma_1, \delta_1$, defined by the equation:

$$\alpha_1^2 + 2\gamma_1^2\delta_1^4\beta_1 - \gamma_1\delta_1^3\beta_1^3 + 40\gamma_1^2\delta_1^3 - 32\gamma_1\delta_1^2\beta_1^2 - 252\gamma_1\delta_1\beta_1 - 2\beta_1^3 - 432\gamma_1 = 0,$$

  and the blowup map $\iota_{2,1} = \iota_2|_{W_1} : W_1 \to U_2$ is:

$$\begin{array}{ccc} W_1 & \xrightarrow{\quad\iota_{2,1}\quad} & U_2 \subset X_1 \\ (\alpha_1, \beta_1, \gamma_1, \delta_1) & \longmapsto & (\gamma_1\alpha_1, \beta_1, \gamma_1, \gamma_1\delta_1). \end{array}$$

  By applying the Jacobian criterion, one finds that the chart W_1 is singular along the (smooth and rational) disjoint curves

$$\widetilde{\widetilde{C}}_{1,1} = \left\{ \alpha_1 = \delta_1\beta_1 + 18 = 108\gamma_1 + \beta_1^3 = 0 \right\}, \quad \widetilde{\widetilde{C}}_{2,1} = \left\{ \alpha_1 = \delta_1\beta_1 + 4 = 32\gamma_1 - \beta_1^3 = 0 \right\}.$$

  In particular, by construction, $\widetilde{\widetilde{C}}_{1,1}$ is a (generically) trivial family of A_2 singularities, while $\widetilde{\widetilde{C}}_{2,1}$ is a (generically) trivial family of A_1 singularities.

  The exceptional locus of the partial resolution $\varepsilon_{2,1} = \varepsilon_2|_{W_1} : W_1 \to X$ is the union of the the following two surfaces:

$$F_1 = \mathrm{Exc}(\iota_{2,1}) = \left\{ \gamma_1 = 2\beta_1^3 - \alpha_1^2 = 0 \right\} \text{ and } \widetilde{E}_1 = \left\{ \delta_1^3\gamma_1 + 2 = \alpha_1^2\delta_1^3 + 64(4 + \beta_1\delta_1)^2 = 0 \right\}$$

Notice that they don't intersect, $\widetilde{E}_1 \cap F_1 = \emptyset$ and that $\widetilde{E}_1$ is the strict transform, via ι_2 of E_2. Moreover, we have

$$\mathrm{Sing}(F_1) = \{\, \alpha_1 = \beta_1 = \gamma_1 = 0 \,\} \text{ and } \mathrm{Sing}(\widetilde{E}_1) = \widetilde{\widetilde{C}}_{2,1}.$$

- **Description of W_2.** The variety W_2 is the hypersurface of $\mathbb{A}^4$, with coordinates $\alpha_2, \beta_2, \gamma_2, \delta_2$, defined by the equation:

$$-2\gamma_2^2\beta_2^3 - 432\gamma_2^3\delta_2 - 252\gamma_2^2\delta_2\beta_2 - 32\gamma_2\delta_2\beta_2^2 - \delta_2\beta_2^3 + 40\gamma_2\delta_2^2 + 2\delta_2^2\beta_2 + \alpha_2^2 = 0$$

and the blowup map $\iota_{2,2} = \iota_2|_{W_2} : W_2 \to U_2$ is:

$$\begin{aligned} W_2 &\xrightarrow{\quad \iota_{2,2} \quad} U_2 \subset X_1 \\ (\alpha_2, \beta_2, \gamma_2, \delta_2) &\longmapsto (\alpha_2\delta_2, \beta_2, \gamma_2\delta_2, \delta_2). \end{aligned}$$

Again by applying the Jacobian criterion, one finds that the chart W_2 is singular along two (smooth and rational) curves, namely

$$\widetilde{\widetilde{C}}_{1,2} = \{\, \alpha_2 = 18\gamma_2 + \beta_2 = 54\gamma_2^2 - \delta_2 = 0 \,\} \text{ and } \widetilde{\widetilde{C}}_{2,2} = \{\, \alpha_2 = 4\gamma_2 + \beta_2 = 2\gamma_2^2 + \delta_2 = 0 \,\}.$$

They intersect at the origin $(0,0,0,0) \in \mathbb{A}^4$ and again, outside some dissident points (e.g. their intersection), they are a family of A_2 singularities and A_1 singularities respectively.

The exceptional locus of the partial resolution $\varepsilon_{2,2} = \varepsilon_2|_{W_2} : W_2 \to X$ is the union of the the following two surfaces:

$$F_2 = \mathrm{Exc}(\iota_{2,2}) = \{\, \delta_2 = 2\gamma_2^2\beta_2^3 - \alpha_2^2 = 0 \,\} \text{ and } \widetilde{E}_2 = \{\, \alpha_2^2 + 64\gamma_2^3(4\gamma_2 + \beta_2)^2 = \delta_2 + 2\gamma_2^2 = 0 \,\}.$$

They intersect along the β_2-axis, i.e. $F_2 \cap \widetilde{E}_2 = \{\, \alpha_2 = \gamma_2 = \delta_2 = 0 \,\} = L$. Moreover, the β_2-axis is also a common singularity for the exceptional divisors and we have:

$$\mathrm{Sing}(F_2) = L \cup \{\, \alpha_2 = \beta_2 = \delta_2 = 0 \,\} \text{ and } \mathrm{Sing}(\widetilde{E}_2) = \widetilde{\widetilde{C}}_{2,2} \cup L.$$

1.8.5 Second change of coordinates

We focus on the chart W_2, where all the complexity of the singular variety X_2 is encoded. Indeed, on the other charts that we have produced, X_2 has only trivial families of DuVal singularities (see Remark 1.8.1). Before to proceed, we act, for the second time, with a change of coordinates φ on $\mathbb{A}^4$. This will produce nicer equations for the involved varieties.

$$\begin{aligned} \mathbb{A}^4 &\xrightarrow{\qquad \varphi \qquad} \mathbb{A}^4 \\ (\alpha_2, \beta_2, \gamma_2, \delta_2) &\longmapsto \left(\tfrac{2}{7}\alpha_2, \tfrac{2}{7}(9\beta_2 - 2\gamma_2), \tfrac{1}{7}(\gamma_2 - \beta_2), \tfrac{\delta_2}{7} - 2\left(\tfrac{\gamma_2 - \beta_2}{7}\right)^2 \right). \end{aligned}$$

The new equation for W_2 is now

$$64\beta_2^5 - 192\beta_2^4\gamma_2 + 192\beta_2^3\gamma_2^2 - 64\beta_2^2\gamma_2^3 - 16\beta_2^3\delta_2 + 88\beta_2^2\gamma_2\delta_2 + 26\beta_2\gamma_2^2\delta_2 + \beta_2\delta_2^2 - 8\gamma_2\delta_2^2 - 7\alpha_2^2 = 0,$$

and we can rewrite the singualrities and the exceptional divisors of W_2 as follows:

$$\widetilde{\widetilde{C}}_{1,2}=\{\,\alpha_2=\gamma_2=\delta_2-8\beta_2^2=0\,\},\quad \widetilde{\widetilde{C}}_{2,2}=\{\,\alpha_2=\beta_2=\delta_2=0\,\}$$

and

$$F_2=\{\,2\delta_2(9\beta_2-2\gamma_2)^3-49\alpha_2^2=7\delta_2-2(\gamma_2-\beta_2)^2=0\,\},\ \widetilde{E}_2=\{\,\delta_2=7\alpha_2^2+64\beta_2^2(\gamma_2-\beta_2)^3=0\,\}$$

Finally, in this new coordinates, we have:

$$\widetilde{E}_2\cap F_2=\{\,\alpha_2=\gamma_2-\beta_2=\delta_2=0\,\}=L$$

and

$$\mathrm{Sing}(F_2)=L\cup\{\,\alpha_2=2\delta_2-7\beta_2^2=9\beta_2-2\gamma_2=0\,\},$$
$$\mathrm{Sing}(\widetilde{E}_2)=\widetilde{\widetilde{C}}_{2,2}\cup L,$$

1.8.6 Blowup of X_2 along $\widetilde{\widetilde{C}}_2$

In practice (see Remark 1.8.1), instead of computing the whole $\iota_3: X_3=\mathrm{Bl}_{\widetilde{\widetilde{C}}_2}\to X_2$, we will perform the blowup $\widetilde{W_2}$ of W_2 along $\widetilde{\widetilde{C}}_{2,2}$. Again $\widetilde{W_2}$ is covered by two affine charts that we will denote by T_1, T_2 and we will describe in next section. We shall denote by $\varepsilon_3: X_3\to X$ the partial resolution map.

Local pictures We describe now $\widetilde{W_2}=T_1\cup T_2$.

- **Description of T_1.** The variety T_1 is the hypersurface of $\mathbb{A}^4$, with coordinates $\lambda_1,\mu_1,\nu_1,\chi_1$, defined by the equation:

$$-16\nu_1^2(25\mu_1-4\nu_1)-(8\mu_1-\chi_1)^2(\mu_1-8\nu_1)-2\nu_1(8\mu_1-\chi_1)(20\mu_1-13\nu_1)+7\lambda_1^2=0$$

  and the blowup map $\iota_{3,1}=\iota_3|_{T_1}: T_1\to W_2$ is:

$$\begin{array}{ccc} T_1 & \xrightarrow{\iota_{2,1}} & W_2\subset X_2 \\ (\lambda_1,\mu_1,\nu_1,\chi_1) & \longmapsto & (\lambda_1\mu_1,\mu_1,\nu_1,\chi_1\mu_1). \end{array}$$

  It is a 3-fold singular along the curve

$$\widetilde{\widetilde{\widetilde{C}}}_{1,1}=\{\,\lambda_1=\nu_1=\chi_1-8\mu_1=0\,\},$$

  which, outside some dissident points is a family of A_2 singularities.

  The exceptional locus of the map $\varepsilon_{3,1}=\varepsilon_3|_{T_1}$ is the union of the following surfaces:

$$G_1=\mathrm{Exc}(\iota_{3,1})=\{\,\mu_1=64\nu_1^3-26\nu_1^2\chi_1+8\nu_1\chi_1^2+7\lambda_1^2=0\,\},\ \widetilde{\widetilde{E}}_1=\{\,\chi_1=7\lambda_1^2-64(\mu_1-\nu_1)^3=0\,\}$$

$$\widetilde{F}_1=\{\,343\lambda_1^2-4(\mu_1-\nu_1)^2(729\mu_1-478\nu_1)-56\nu_1\chi_1(23\nu_1-7\chi_1)=7\mu_1\chi_1-2(\mu_1-\nu_1)^2=0\,\}.$$

They intersect as follows:

$$G_1 \cap \widetilde{\widetilde{E}}_1 = \left\{ \mu_1 = \chi_1 = 64\nu_1^3 + 7\lambda_1^2 = 0 \right\},\ G_1 \cap \widetilde{F}_1 = \left\{ \lambda_1 = \mu_1 = \nu_1 = 0 \right\} = Z_1,$$

$$\widetilde{F}_1 \cap \widetilde{\widetilde{E}}_1 = \left\{ \lambda_1 = \mu_1 - \nu_1 = \chi_1 = 0 \right\} = \widetilde{L},$$

and their singularities are:

$$\mathrm{Sing}(\widetilde{\widetilde{E}}_1) = \widetilde{L},\ \mathrm{Sing}(G_1) = \left\{ \lambda_1 = \mu_1 = \nu_1 = \chi_1 = 0 \right\} = \{p\},$$

$$\mathrm{Sing}(\widetilde{F}_1) = \widetilde{L} \cup \left\{ \lambda_1 = 9\mu_1 - 2\nu_1 = 7\nu_1 - 9\chi_1 = 0 \right\}.$$

Notice that, since the equations of G_1 can be put, via an appropriate change of coordinates, in the form

$$\mu_1 = \lambda_1^2 - \nu_1(\nu_1^2 - \mu_1^2) = 0,$$

the point p is a DuVal singularity of type D_4.

- **Description of T_2.** The variety T_2 is the hypersurface of $\mathbb{A}^4$, with coordinates $\lambda_2, \mu_2, \nu_2, \chi_2$, defined by the equation:

$$64\mu_2^5\chi_2^3 - 192\mu_2^4\nu_2\chi_2^2 + 192\mu_2^3\nu_2^2\chi_2 - 64\mu_2^2\nu_2^3 - 16\mu_2^3\chi_2^2 + 88\mu_2^2\nu_2\chi_2 + 26\mu_2\nu_2^2 - 7\lambda_2^2 + \mu_2\chi_2 - 8\nu_2 = 0$$

and the blowup map $\iota_{3,2} = \iota_3|_{T_2} : T_2 \to W_2$ is:

$$\begin{array}{ccc} T_2 & \xrightarrow{\iota_{3,2}} & W_2 \subset X_2 \\ (\lambda_2, \mu_2, \nu_2, \chi_2) & \longmapsto & (\lambda_2\chi_2, \mu_2\chi_2, \nu_2, \chi_2). \end{array}$$

The variety T_2 is singular along the curve

$$\widetilde{\widetilde{\widetilde{C}}}_{1,2} = \left\{ \lambda_2 = \nu_2 = 8\mu_2^2\chi_2 - 1 = 0 \right\}.$$

In this chart, we don't see the strict transform of $E \subset X_1$ and the exceptional locus is the union of the following surfaces

$$G_2 = \mathrm{Exc}(\iota_{3,2}) = \left\{ \chi_2 = 64\mu_2^2\nu_2^3 - 26\mu_2\nu_2^2 + 7\lambda_2^2 + 8\nu_2 = 0 \right\}$$

$$\widetilde{F}_2 = \left\{ 2(\mu_2\chi_2 - \nu_2)^2 - 7\chi_2 = 280\mu_2^2\nu_2\chi_2 - 182\mu_2\nu_2^2 - 7\lambda_2^2 + 729\mu_2\chi_2 - 8\nu_2 = 0 \right\}$$

which intersect along:

$$\widetilde{F}_2 \cap G_2 = \left\{ \lambda_2 = \nu_2 = \chi_2 = 0 \right\} = Z_2.$$

Finally, the singularities of the prime divisors are:

$$\mathrm{Sing}(\widetilde{F}_2) = \left\{ \lambda_2 = 7\mu_2\nu_2 - 9 = 7\mu_2^2\chi_2 - 2 = 0 \right\},\ \mathrm{Sing}(G_2) = \emptyset.$$

1.8.7 Third change of coordinates

We focus now on the chart T_1 and we act for the third time with a biholomorphism ϑ. Again, we recall (see Remark 1.8.1), that this is enough to understand the whole partial resolution.

The change of coordinates ϑ of $\mathbb{A}^4$ is defined as follows:

$$\begin{aligned} \mathbb{A}^4 &\xrightarrow{\qquad\quad \vartheta \qquad\quad} \mathbb{A}^4 \\ (\lambda_1,\mu_1,\nu_1,\chi_1) &\longmapsto (\lambda_1,\mu_1/2,\nu_1/2,\chi_1+4\mu_1). \end{aligned}$$

and the new equation for T_1 is:

$$\mu_1(10\nu_1-\chi)^2-\nu_1(16\nu_1^2-13\nu_1\chi_1+8\chi_1^2)-14\lambda_1^2=0.$$

The new equations of the objects in the previous section, with respect to the new coordinates, are:

$$\mathrm{Sing}(T_1)=\tilde{\tilde{\tilde{C}}}_{1,1}=\left\{\lambda_1=\nu_1=\chi_1=0\right\},$$

$$G_1=\left\{\mu_1=16\nu_1^3-13\nu_1^2\chi_1+8\nu_1\chi_1^2+14\lambda_1^2=0\right\},\ \tilde{\tilde{E}}_1=\left\{4\mu_1+\chi_1=(4\nu_1+\chi_1)^3+56\lambda_1^2=0\right\},$$

$$\widetilde{F}_1=\left\{27\mu_1^2+2\mu_1\nu_1-\nu_1^2+7\mu_1\chi_1=\mu_1(10\nu_1-\chi_1)^2-16\nu_1^3+13\nu_1^2\chi_1-8\nu_1\chi_1^2-14\lambda_1^2=0\right\},$$

$$G_1\cap\tilde{\tilde{E}}_1=\left\{\chi_1,=\mu_1=8\nu_1^3+7\lambda_1^2=0\right\},\ G_1\cap\widetilde{F}_1=\left\{\lambda_1=\mu_1=\nu_1=0\right\}=Z_1,$$

$$\widetilde{F}_1\cap\tilde{\tilde{E}}_1=\left\{\lambda_1=\mu_1-\nu_1=\chi_1+4\mu_1=0\right\}=\widetilde{L},$$

$$\mathrm{Sing}(\tilde{\tilde{E}}_1)=\widetilde{L},\ \mathrm{Sing}(G_1)=\left\{\lambda_1=\mu_1=\nu_1=\chi_1=0\right\}=\{p\},$$

$$\mathrm{Sing}(\widetilde{F}_1)=\widetilde{L}\cup\left\{\lambda_1=2\chi_1+\nu_1=4\chi_1+9\mu_1=0\right\}.$$

Recall that the point p is a DuVal singularity of type D_4.

1.8.8 The last blowup

The last step consists of the blowup of X_3 with centre $\tilde{\tilde{\tilde{C}}}_1=\tilde{\tilde{\tilde{C}}}_{1,1}\cup\tilde{\tilde{\tilde{C}}}_{1,2}\subset X_3$. As usual, we will focus on the chart where all the mutual intersections of the exceptional divisors occur. In practice, we will cover $\mathrm{Bl}_{\tilde{\tilde{\tilde{C}}}_{1,1}}T_1$ with two charts S_1 and S_2.

Local Pictures Both charts are smooth, and we describe them as hypersurfaces $S_i\subset\mathbb{A}^4$, with coordinates x_i,y_i,z_i,t_i for $i=1,2$. In the following table, that lists their main properties, we will omit the indices of the variables for the sake of readability.

CHARTS	
S_1	S_2
EQUATIONS	
$-y(t-10)^2+z(8t^2-13t+16)+14x^2=0$	$-y(10z-1)^2+16z^3t-13z^2t+8tz+14x^2=0$
BLOWUP MAPS	
$(x,y,z,t)\mapsto(xz,y,z,tz)$	$(x,y,z,t)\mapsto(xt,y,zt,t)$
EXCEPTIONAL LOCI	
$H_1=\mathrm{Exc}(\iota_{4,1})=\begin{cases} z=0\\ y(t-10)^2-14x^2=0\end{cases}$ $\widetilde{G}_1=\begin{cases} y=0\\ 8zt^2+14x^2-13zt+16z=0\end{cases}$ $\tilde{\tilde{\tilde{E}}}_1=\begin{cases} zt+4y=0\\ z(t+4)^3+56x^2=0\end{cases}$ $\tilde{\tilde{F}}_1=\begin{cases} \frac{y(t-10)^2-(8t^2-13t+16)z}{14}-x^2=0\\ (z-y)^2-7y(zt+4y)=0\end{cases}$	$H_2=\mathrm{Exc}(\iota_{4,2})=\begin{cases} t=0\\ y(10z-1)^2-14x^2=0\end{cases}$ $\widetilde{G}_2=\begin{cases} y=0\\ 16z^3t-13z^2t+14x^2+8zt=0\end{cases}$ $\tilde{\tilde{\tilde{E}}}_2=\begin{cases} 4y+t=0\\ t(4z+1)^3+56x^2=0\end{cases}$ $\tilde{\tilde{F}}_2=\begin{cases} \frac{y(10z-1)^2-zt(16z^2-13z+8)}{14}-x^2=0\\ (zt-y)^2-7y(4y+t)=0\end{cases}$
SINGULARITIES	
$\mathrm{Sing}(H_1)=\{x=z=t-10=0\}=D_1$ $\mathrm{Sing}(\widetilde{G}_1)=\{p_1,p_2\}$ two A_1 singularities $\mathrm{Sing}(\tilde{\tilde{\tilde{E}}}_1)=\{x=y-z=t+4=0\}=\tilde{\tilde{L}}_1$ $\mathrm{Sing}(\tilde{\tilde{F}}_1)=\tilde{\tilde{L}}_1\cup\{x=y=z=0\}\cup$ $\{x=9y-2z=2t+1=0\}$	$\mathrm{Sing}(H_2)=\{x=t=10z-1=0\}=D_2$ $\mathrm{Sing}(\widetilde{G}_2)=\{p_1,p_2\}\cup\{p_3\}$ three A_1 singularities $\mathrm{Sing}(\tilde{\tilde{\tilde{E}}}_2)=\{x=4y+t=4z+1=0\}=\tilde{\tilde{L}}_2$ $\mathrm{Sing}(\tilde{\tilde{F}}_2)=\tilde{\tilde{L}}_1\cup\{x=y=t=0\}\cup$ $\{x=9y+4t=z+2=0\}$
INTERSECTIONS	
$H_1\cap\widetilde{G}_1=\{x=y=z=0\}=M_1$ $H_1\cap\tilde{\tilde{F}}_1=M_1$ $H_1\cap\tilde{\tilde{\tilde{E}}}_1=M_1$ $\widetilde{G}_1\cap\tilde{\tilde{F}}_1=M_1$ $\widetilde{G}_1\cap\tilde{\tilde{\tilde{E}}}_1=M_1\cup\{y=t=7x^2+8z=0\}$ $\tilde{\tilde{F}}_1\cap\tilde{\tilde{\tilde{E}}}_1=\tilde{\tilde{L}}_1\cup M_1$	$H_2\cap\widetilde{G}_2=\{x=y=t=0\}=M_2$ $H_2\cap\tilde{\tilde{F}}_2=M_2$ $H_2\cap\tilde{\tilde{\tilde{E}}}_2=M_2$ $\widetilde{G}_2\cap\tilde{\tilde{F}}_2=M_2\cup\{x=y=z=0\}$ $\widetilde{G}_2\cap\tilde{\tilde{\tilde{E}}}_2=M_2$ $\tilde{\tilde{F}}_2\cap\tilde{\tilde{\tilde{E}}}_2=\tilde{\tilde{L}}_2\cup M_2$

Remark 1.8.2. Both charts S_1,S_2 are isomorphic to $\mathbb{A}^3$. For instance, one can check that the map

$$\begin{aligned} \mathbb{A}^3 &\longrightarrow S_1\\ (a,b,c) &\longmapsto \left(a,\tfrac{12}{7^3}a^2c+\tfrac{18}{7^3}bc^2+\tfrac{47}{7^22}a^2+\tfrac{27}{2^27}bc+\tfrac{9}{2}b,\tfrac{3}{7^32}a^2c+\tfrac{9}{2^27^3}bc^2-\tfrac{1}{49}a^2,c+10\right). \end{aligned}$$

is an isomorphism.

Global picture The crepant resolution $X_4=\mathrm{Bl}_{\tilde{\tilde{C}}_1}X_3$ has four exceptional prime divisors, each coming from a blowup. In particular, $\tilde{\tilde{\tilde{E}}}$, $\tilde{\tilde{F}}$, $\widetilde{G}$ and H are the respective (strict transforms of the) exceptional divisors of $\varepsilon_1,\iota_2,\iota_3,\iota_4$.

We describe now these divisors. In Figure 1.25, a real version of the exceptional divisor of some open subset of $S_1 \cup S_2$ is depicted. We will choose the notation consistently with respect to the above table.

- The exceptional divisor $\tilde{\tilde{\tilde{E}}}$ is the strict transform, via $\iota_4 \circ \iota_3 \circ \iota_2 : X_4 \to X_1$, of the exceptional divisor E. It is singular along the curve $\tilde{\tilde{L}}$. Locally, near $\tilde{\tilde{L}} \setminus \{p\}$, the divisor $\tilde{\tilde{\tilde{E}}}$ is a trivial family of cusps. The restriction of the resolution map ε_4 to $\tilde{\tilde{\tilde{E}}}$,
$$\varepsilon_4|_{\tilde{\tilde{\tilde{E}}}} : \tilde{\tilde{\tilde{E}}} \to C_2,$$
is, outside the singularity of C_2, a trivial family of projective singular non-degenerate conics (pairs of incident lines). Finally, $\tilde{\tilde{\tilde{E}}}$ intersect $\tilde{\tilde{F}}$ along the curves $\tilde{\tilde{L}}$ and M.

- The exceptional divisor $\tilde{\tilde{F}}$ is the strict transform, via $\iota_4 \circ \iota_3 : X_4 \to X_2$, of the exceptional divisor F. It is singular along the curves $\tilde{\tilde{L}}$, M, N. Locally, near $\tilde{\tilde{L}} \setminus \{p\}$, the divisor $\tilde{\tilde{F}}$ is a trivial family of nodes, while, near $(M \cup N) \setminus \{p\}$, it is a family of cusps. The restriction of the resolution map ε_4 to $\tilde{\tilde{F}}$,
$$\varepsilon_4|_{\tilde{\tilde{F}}} : \tilde{\tilde{F}} \to \{0\},$$
is constant.

- The exceptional divisor $\widetilde{G}$ is the strict transform, via $\iota_4 : X_4 \to X_3$, of the exceptional divisor G. The restriction of ι_4, $\iota_4|_{\widetilde{G}} : \widetilde{G} \to G$ is a partial resolution of the D_4 singularity of G. In particular $\iota_4|_{\widetilde{G}}$ agree with the blowup $\mathrm{Bl}_{\mathrm{Sing}(G)} G$ and it is well known that $\widetilde{G}$ has one irreducible exceptional divisor M over which three A_1 singularities lie, namely p_1, p_2, p_3. Now, the restriction
$$\varepsilon_4|_{\widetilde{G}} : \widetilde{G} \to C_2$$
is, outside the singularity of C_2 a trivial family of projective lines, while, as expected, again outside the singularity of C_2, the restriction
$$\varepsilon_4|_{\widetilde{G} \cup E} : \widetilde{G} \to C_2$$
is a family of chains of three projective lines. Finally, $\widetilde{G}$ intersects $\tilde{\tilde{\tilde{E}}}$ along the curves M and V, it intersect $\tilde{\tilde{F}}$ along the curves M and $\widetilde{Z}$ and it intersects the divisor H along M.

- The divisor H is the exceptional divisor of the map ι_4. It is a Whitney umbrella, i.e. it is isomorphic to the surface $A = \left\{ ((\alpha, \beta), [u : v]) \in \mathbb{A}^2 \times \mathbb{P}^1 \,\middle|\, \alpha^2 u = \beta^2 v \right\}$. It intersects all the others exceptional divisors along the curve M and it is singular along the curve D. Finally, the restriction
$$\varepsilon_4|_H : H \to C_1$$
is, outside the singularity of C_1 a trivial family of nodes (incident lines).

Conclusions. Now that we have this explicit description of a resolution X_4, we can study the sequence of flops that connect X_4 and the crepant resolution $\overline{X}_4$ built by Markushevich in [50]. Moreover, since we know from [71] that Craw–Ishii's conjecture (Conjecture 1.0.24) is true, we can look for the chamber $C \subset \Theta^{\text{gen}}$ such that $X_4 \cong \mathscr{M}_C$. We leave this study for the future, and here we simply give the correspondence between the exceptional divisors of X_4 and those of $\overline{X}_4$. The existence of this correspondence is a consequence of the fact that all crepant resolutions are isomorphic in codimension 1.

The correspondence goes as follows.

- The divisor $\widetilde{\widetilde{\widetilde{E}}}$ corresponds to the divisor of $\overline{X_4}$ coming from the 3-rd blowup in [50],
- the divisor $\widetilde{\widetilde{F}}$ corresponds to the divisor of $\overline{X_4}$ coming from the 2-nd blowup in [50],
- the divisor $\widetilde{G}$ corresponds to the divisor of $\overline{X_4}$ coming from the 4-th blowup in [50],
- the divisor H corresponds to the divisor of $\overline{X_4}$ coming from the 1-st blowup in [50].

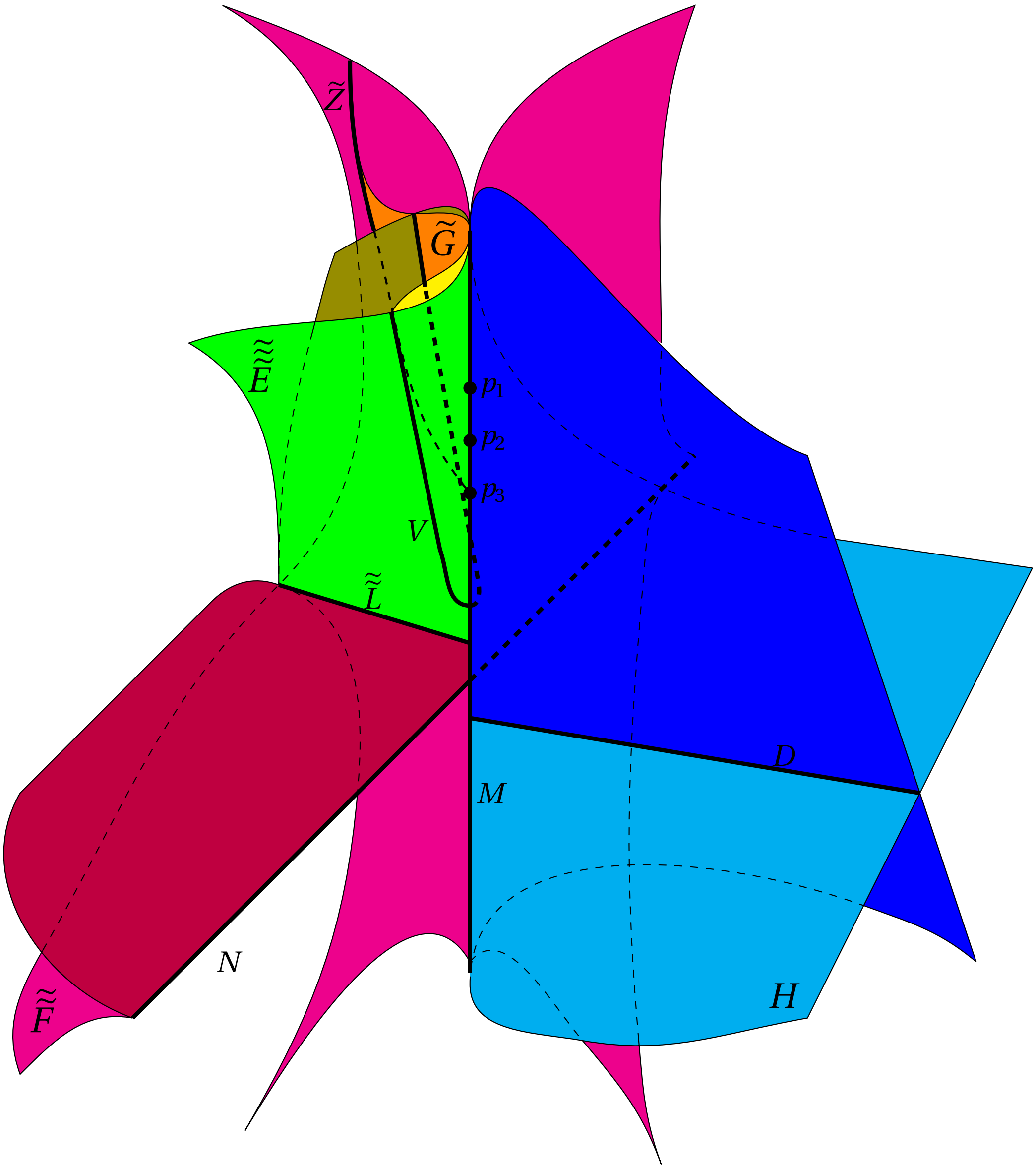


Figure 1.25. A qualitative real version of $\operatorname{Exc}(\varepsilon_4|_V)$ for some $V \subset S_1 \cup S_2$.

Chapter 2

On the Behrend function and the blowup of some fat points

2.0 Overview of the topic

Let X be a scheme of finite type over the field $\mathbb{C}$. One of the (arguably few) intrinsic geometric objects attached to X is a certain cone stack $\mathfrak{C}_X \to X$, called the *intrinsic normal cone*, constructed by Behrend–Fantechi in their seminal work [6]. This construction was a breakthrough in enumerative geometry, for it opened the way to a rigorous definition of a cascade of invariants that have been of central importance in algebraic geometry ever since: Gromov–Witten invariants, Donaldson–Thomas invariants, stable pair invariants to name a few.

A few years after the intrinsic normal cone was born, Behrend [5] proved that any scheme X of finite type over $\mathbb{C}$ carries a canonical constructible function

$$\nu_X : X(\mathbb{C}) \to \mathbb{Z},$$

defined (cf. Section 2.1) as the local Euler obstruction of a canonical cycle $\mathfrak{c}_X \in Z_*(X)$ called the *signed support of the intrinsic normal cone*. This function is universally referred to as the *Behrend function*, and it has the following remarkable property: whenever X is proper and carries a perfect symmetric obstruction theory (in the sense of Behrend–Fantechi [7]), the degree of the virtual fundamental class $[X]^{\mathrm{vir}} \in A_0(X)$ attached to the obstruction theory agrees with the weighted Euler characteristic of X,

$$\chi(X, \nu_X) = \sum_{n \in \mathbb{Z}} n\chi(\nu_X^{-1}(n)),$$

the 'weight' being precisely the Behrend function [5]. This result allowed algebraic geometers to compute a large number of enumerative invariants, previously inaccessible, attached to moduli spaces X satisfying the assumptions of Behrend's theorem. See e.g. [67, 55] for some background and references related to this subject.

Of course, knowing the precise values of the Behrend function is a more refined information than knowing just the weighted Euler characteristic. Unfortunately, the Behrend function is quite elusive. We refer the reader to the original papers [5, 7] for its main properties, some of which are recalled in Section 2.1. First of all, it is an *invariant of singularities*, in the sense that it pulls back along étale maps; in particular, it is sensitive to the scheme structure. It satisfies $\nu_X(p) = (-1)^{\dim_p X}$ if p is a smooth point of X. When X is a critical locus, i.e. a scheme of the form $V(\mathrm{d}f) \subset U$, for some regular function f on a smooth scheme U, the function ν_X agrees with the *Milnor function* of (U, f). See Example 2.2.1 for more details. Not much more is known about the Behrend function in general. See Open problem G for a hard open problem in Donaldson–Thomas theory, related to the Behrend function and also to the geometry of Quot schemes.

As mentioned above, the Behrend function has a crucial role in those enumerative theories where the moduli spaces involved carry a symmetric obstruction theory. This is for instance the case of Donaldson–Thomas theory (DT theory, for short), an enumerative theory designed to 'count' sheaves on smooth 3-folds [69]. If X is a moduli space of stable sheaves on a projective Calabi–Yau 3-fold, then the expected dimension of X is 0. When X really has dimension 0, it is equal to a disjoint union $X_1 \amalg \cdots \amalg X_e$ where each X_i is a fat point, the main object of study in this thesis. That is, we have $X = X_1 \amalg \cdots \amalg X_e$ where X_i is a fat point. If X_i is reduced for all i, then the DT invariant is just the number of points, namely e. But in the general case, the DT invariant is

$$\chi(X, \nu_X) = \sum_{i=1}^{e} \nu_{X_i},$$

which is one motivation for the interest in the computation of the Behrend number of a fat point. Even though a moduli space as above is rarely 0-dimensional, there are examples where this actually happens, see e.g. [69, Thm. 3.55 and § 4], but also [67, Ex. 8.1] and [68, Thm. 1.1].

We conclude the introduction with a challenging open problem in DT theory.

Open Problem G. Fix integers $r \geq 1$ and $n \geq 0$, and let $\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n)$ be the Quot scheme parametrising length n quotients of the trivial sheaf $\mathscr{O}^{\oplus r}$ on $\mathbb{A}^3$, a key character in DT theory [62, 22]. As proved in [4] (see [7] for the $r = 1$ case), there is an identity

$$\chi(\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n), \nu) = (-1)^{rn} \chi(\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n)),$$

and the value of the Behrend function at a torus-fixed quotient $p = [\mathscr{O}^{\oplus r} \twoheadrightarrow T]$, with respect to the natural $(\mathbb{C}^\times)^{3+r}$-action on the Quot scheme, is

$$\nu(p) = (-1)^{rn}.$$

However, it is not known whether ν is *constantly* equal to $(-1)^{rn}$. Its constancy would show that the Quot scheme mentioned above is generically reduced, which is currently unknown even for $r = 1$, i.e. in the case of the Hilbert scheme of points $\mathrm{Hilb}^n(\mathbb{A}^3)$.

In dimension $N > 3$, the question of the reducedness of $\mathrm{Quot}_{\mathbb{A}^N}(\mathscr{O}^{\oplus r}, n)$ has already been answered in the negative: see [43, § 6.5] for an example of a generically nonreduced component of $\mathrm{Quot}_{\mathbb{A}^N}(\mathscr{O}^{\oplus r}, 8)$, where $r > 3$. See also the recent work of Szachniewicz [66] for a proof of the fact that $\mathrm{Hilb}^{13}(\mathbb{A}^6)$ is nonreduced.

2.1 Definition and main properties of the Behrend function

Let X be a scheme of finite type over $\mathbb{C}$, and let $\mathrm{Con}(X)$ be the abelian group of ($\mathbb{Z}$-valued) constructible functions on X. In [5], Behrend constructs a canonical constructible function

$$\nu_X : X(\mathbb{C}) \to \mathbb{Z},$$

nowadays referred to as the 'Behrend function' of X. It has been proven a powerful tool in enumerative geometry, mainly because of the following remarkable property: whenever X is proper and carries a symmetric perfect obstruction theory in the sense of Behrend–Fantechi [7], one has an identity

$$\int_{[X]^{\mathrm{vir}}} 1 = \chi(X, \nu_X),$$

where the left hand side is the degree of $[X]^{\mathrm{vir}} \in A_0(X)$, the virtual fundamental class attached to the obstruction theory, and the right hand side is the *weighted Euler characteristic* of X, the 'weight' being ν_X. Explicitly, for a constructible function $\gamma \in \mathrm{Con}(X)$, one defines

$$\chi(X, \gamma) = \sum_{m \in \mathbb{Z}} m \chi(\gamma^{-1}(m)).$$

The Behrend function of a scheme X is defined as

$$\nu_X = \mathrm{Eu}(\mathfrak{c}_X),$$

where $\mathrm{Eu}\colon Z_*(X) \xrightarrow{\sim} \mathrm{Con}(X)$ is the *local Euler obstruction*, an isomorphism from cycles on X to constructible functions on X, and $\mathfrak{c}_X$ is a canonical cycle attached to X. The definition of Eu is recalled in [5] and is classical; we refer the reader to Jiang's work [44] for more details on local Euler obstruction (both in algebraic and analytic setting) and the Behrend function. Here we recall how the cycle $\mathfrak{c}_X$ is defined. Suppose $(f\colon U \to X, \iota\colon U \hookrightarrow M)$ is a local embedding for X, i.e. f is an étale morphism of $\mathbb{C}$-schemes, and ι is a closed immersion into a smooth $\mathbb{C}$-scheme M. Let

$$\pi\colon C_{U/M} \to U$$

be the normal cone of this immersion. Note that $C_{U/M}$ is of pure dimension $\dim M$. One can form the cycle

$$\mathfrak{c}_{U/M} = \sum_{D \subset C_{U/M}} (-1)^{\dim \pi(D)} \mathrm{mult}_D(C_{U/M})[\pi(D)] \in Z_*(U),$$

where the sum ranges over all irreducible components D of $C_{U/M}$, and $\mathrm{mult}_D(C_{U/M})$ denotes the *geometric multiplicity* of the irreducible component D, namely the length

$$\mathrm{mult}_D(C_{U/M}) = \mathrm{length}_{\mathscr{O}_{C_{U/M},D}}(\mathscr{O}_{C_{U/M},D})$$

of the artinian ring $\mathscr{O}_{C_{U/M},D}$ viewed as a module over itself, see e.g. [65, Tag 0DR4]. The cycles $\mathfrak{c}_{U/M}$ naturally glue together along local embeddings to give a cycle $\mathfrak{c}_X \in Z_*(X)$, i.e. there exists a unique global cycle $\mathfrak{c}_X$ such that if $(f\colon U \to X, \iota\colon U \hookrightarrow M)$ is a local embedding as above, one has $\mathfrak{c}_X|_U = \mathfrak{c}_{U/M}$.

When X has a global embedding $\iota\colon X \hookrightarrow M$ inside a smooth scheme M (e.g. when X is quasiprojective), we can use the local embedding (id_X, ι) and compute directly

$$\nu_X = \mathrm{Eu}(\mathfrak{c}_X) = \sum_{D \subset C_{X/M}} (-1)^{\dim \pi(D)} \mathrm{mult}_D(C_{X/M}) \mathrm{Eu}([\pi(D)]). \tag{2.1.1}$$

Thus, when X is a fat point, say with embedding dimension N, we have a closed immersion of X inside $M = \mathbb{A}^N$ and Equation (2.1.1) becomes

$$\nu_X = \sum_{D \subset C_{X/M}} \mathrm{mult}_D(C_{X/M}), \tag{2.1.2}$$

because the local Euler obstruction of $[X]$ is equal to 1 and $\dim \pi(D) = 0$.

Notation 2.2. Occasionally, for the sake of readability, if $R = \mathbb{C}[x_1, \dots, x_N]/I$ defines a fat point $X = \mathrm{Spec}\, R \subset \mathbb{A}^N$, we shall write ν_R instead of ν_X, referring to the latter as the *Behrend number* of I, since $\mathrm{Spec}\, R$ has only one point.

The Behrend function also has the following remarkable property, which has been exploited several times for computations in Donaldson–Thomas theory, see e.g. [7, 4].

Example 2.2.1 ([56, Cor. 2.4 (iii)])**.** When X is a *critical locus*, i.e. $X = V(\mathrm{d}f)$ is the zero scheme of an exact 1-form on a smooth scheme M, one has the relation

$$\nu_X(p) = (-1)^{\dim M}(1-\chi(\mathsf{MF}_{f,p})), \tag{2.2.1}$$

where $\mathsf{MF}_{f,p}$ is the Milnor fibre of f at $p \in X$. The right hand side is, by definition, the value of the *Milnor function* attached to (U, f). The above situation includes the important case $f = 0 \in \mathscr{O}_M(M)$, which yields $X = M$ and the formula $\nu_M(p) = (-1)^{\dim_p M}$. So the Behrend function of a smooth point of a scheme is always ± 1.

Ideally, it would be nice to compute the Behrend function of an arbitrary 0-dimensional $\mathbb{C}$-scheme. It is of course enough to perform the computation for fat points, since a finite scheme is a disjoint union of fat points. The Behrend function of a fat point X is the constant given by Equation (2.1.2), so our goal is to compute this constant exploiting such relation in a large number of cases.

If X is a (proper) moduli space of sheaves on a Calabi–Yau 3-fold of dimension equal to the expected dimension, namely 0, then $X = X_1 \amalg \cdots \amalg X_e$ is a disjoint union of fat points X_i, and the non-reducedness of X_i is a shadow of the existence of obstructed deformations for the object parametrised by $X_{i,\mathrm{red}} \hookrightarrow X$. Even though there is in general no control on these obstructions, one can compute the Donaldson–Thomas invariant of X as the integer $\nu_{X_1} + \cdots + \nu_{X_e}$.

2.2.1 A formula for the Behrend function in terms of blowup

Let $X = \operatorname{Spec} R$ be a fat point over $\mathbb{C}$, and let $X \hookrightarrow U$ be a closed immersion into a smooth affine scheme U. Let $I \subset \mathscr{O}_U$ be the ideal defining this inclusion, so that $R \cong \mathscr{O}_U/I$, and let $C = C_{X/U} = \operatorname{Spec}\left(\bigoplus_{d\geq 0} I^d/I^{d+1}\right)$ be the normal cone of $X \hookrightarrow U$. As in Equation (2.1.2), we have

$$\nu_X = \sum_{D\subset C} \mathrm{length}_{\mathscr{O}_{C,D}}\left(\mathscr{O}_{C,D}\right), \tag{2.2.2}$$

where the sum runs over all irreducible components D of C. This sum does not depend on the particular embedding $X \hookrightarrow U$ we picked. If $N = \dim U$, then we know that C is purely N-dimensional, but we do not know how many irreducible components it has in general; however, in the case where $I \subset \mathbb{C}[x, y]$ is a normal monomial ideal, the number of components of the normal cone to the fat point $X = V(I) \hookrightarrow \mathbb{A}^2$ can be computed via Theorem 2.6.12. Note that a natural choice for U is the affine space $\mathbb{A}^N$, where $N = \dim_{\mathbb{C}}(\mathfrak{m}_R/\mathfrak{m}_R^2)$ is the embedding dimension of $X = \operatorname{Spec} R$.

One first observation, towards the computation of ν_X via Equation (2.2.2), is that the projective cone $P(C)$ sits in the projective completion $P(C \oplus \mathbb{1})$ as the divisor 'at infinity', with open *dense* complement equal to C. Hence we may rewrite Equation (2.2.2) as

$$\nu_X = \sum_{D\subset C} \mathrm{length}_{\mathscr{O}_{P(C\oplus\mathbb{1}),P(D\oplus\mathbb{1})}}\left(\mathscr{O}_{P(C\oplus\mathbb{1}),P(D\oplus\mathbb{1})}\right). \tag{2.2.3}$$

We notice that $P(C \oplus \mathbb{1})$ is the exceptional divisor of a blowup, just as $P(C)$ is. We consider the embedding $X \hookrightarrow \mathbb{A}^N \hookrightarrow \mathbb{A}^N \times \mathbb{A}^1 = M$, where the second map is induced by the inclusion of the origin 0 in $\mathbb{A}^1$. Then, we have an identity

$$E_X M = P(C \oplus \mathbb{1}) \subset \mathrm{Bl}_X M,$$

so by Equation (2.2.3) we have to determine the geometric multiplicities of the irreducible components of the exceptional divisor $E_X M$.

Equation (2.2.3) will be used to explicitly compute the Behrend function of curvilinear schemes (see Example 2.2.5). However, for most of Chapter 2 the key relation that will be exploited is the one contained in the following lemma.

Lemma 2.2.2. *Let $X \subset \mathbb{A}^N$ be a fat point, with normal cone $C = C_{X/\mathbb{A}^N} = \operatorname{Spec} S$ and associated projective cone $P = P(C) = \mathbb{P}S$. Then the association $D \mapsto P(D)$ is a bijective correspondence between irreducible components of C and irreducible components of P, and there is an identity*

$$\nu_X = \sum_{D \subset C} \operatorname{length}_{\mathscr{O}_{P(C),P(D)}} \left(\mathscr{O}_{P(C),P(D)}\right).$$

In 'blowup language', this can be rephrased as

$$\nu_X = \sum_{D \subset C} \operatorname{mult}_{P(D)} E_X \mathbb{A}^N.$$

Proof. For a general cone $C = \operatorname{Spec} S$ over an affine scheme $X = \operatorname{Spec} R$, the irreducible components $D \subset C$ are themselves cones (over subvarieties of X), each of which is given as $D = V(\mathfrak{p}) = \operatorname{Spec}(S/\mathfrak{p})$, where $\mathfrak{p} \subset S$ is a homogeneous minimal prime ideal. Thus $D \mapsto P(D)$ is a bijection.

If $X \subset \mathbb{A}^N$ is a fat point with normal cone C as in the statement, and $D = V(\mathfrak{p}) \subset C$ is an irreducible component, the local ring $S_{\mathfrak{p}}$ is artinian, as well as the homogeneous localisation $S_{(\mathfrak{p})}$, and we have

$$\begin{aligned}
\operatorname{length}_{\mathscr{O}_{C,D}}\left(\mathscr{O}_{C,D}\right) &= \operatorname{length}_{S_{\mathfrak{p}}}\left(S_{\mathfrak{p}}\right) \\
&= \operatorname{length}_{S_{(\mathfrak{p})}}\left(S_{(\mathfrak{p})}\right) \\
&= \operatorname{length}_{\mathscr{O}_{P(C),P(D)}}\left(\mathscr{O}_{P(C),P(D)}\right),
\end{aligned}$$

which by Equation (2.2.2) implies the formula for ν_X. □

2.2.2 The Behrend function of the easiest fat points

We conclude this section with some examples of computation of ν_X for X a fat point.

Example 2.2.3 (Critical loci)**.** Let $X = \operatorname{Spec} R$ be a fat point that is also a critical locus, i.e. the zero locus of an exact 1-form $\mathrm{d}f$, for some function $f \in \mathscr{O}_U(U)$ on a smooth scheme U. In particular, R is equal to the Jacobian ring attached to (U, f), whose dimension as a $\mathbb{C}$-vector space is by definition the Milnor number μ_f. In this case, one has

$$\nu_X = \operatorname{length}(X).$$

Indeed, since X_{red} is just one point, $X \hookrightarrow U$ is isolated and so Equation (2.2.1) holds, giving

$$\nu_X = (-1)^{m+1}(1 - \chi(\mathsf{MF}_f)),$$

where $m+1$ is the complex dimension of U and where MF_f has the same homotopy type of a bouquet of μ_f spheres $\mathbb{S}^m \subset \mathbb{R}^{m+1}$. This implies

$$\chi(\mathsf{MF}_f) = \mu_f \cdot (1+(-1)^m) - (\mu_f - 1) = 1 + (-1)^m \mu_f,$$

which indeed gives

$$\nu_X = (-1)^{m+1}(1-(1+(-1)^m\mu_f)) = \mu_f = \dim_{\mathbb{C}}(R) = \operatorname{length}(X).$$

Example 2.2.4 (Local complete intersections)**.** Let $X \subset \mathbb{A}^N$ be fat point that is also a local complete intersection subscheme. Then $C_{X/\mathbb{A}^N} = N_{X/\mathbb{A}^N}$ is the total space of a vector bundle over X of rank N. Thus $P(C_{X/\mathbb{A}^N})$ is a $\mathbb{P}^{N-1}$-bundle over X, and as such it is irreducible, with multiplicity equal to length(X). So

$$\nu_X = \operatorname{length}(X).$$

Example 2.2.5 (Curvilinear schemes)**.** Fix an integer $n>0$ and consider the curvilinear scheme $X_n = \operatorname{Spec}\mathbb{C}[t]/t^n$. Then $\nu_{X_n} = n$ follows by both Example 2.2.3 and Example 2.2.4. We first confirm this formula by means of Equation (2.2.2), as follows: for every $d \geq 0$, the d-th graded piece of the coordinate ring of $C_{X_n/\mathbb{A}^1}$ is isomorphic to $R_n = \mathbb{C}[t]/t^n$ as an R_n-module: if $I = (t^n)$, then

$$I^d/I^{d+1} = \left\langle t^{nd}, t^{nd+1}, \ldots, t^{nd+n-1}\right\rangle_{\mathbb{C}} \cong R_n.$$

Thus

$$\bigoplus_{d\geq 0} I^d/I^{d+1} = \bigoplus_{d\geq 0} R_n \cdot z^d = \mathbb{C}[t,z]/t^n,$$

proving that

$$C_{X_n/\mathbb{A}^1} = \mathbb{A}^1 \times X_n,$$

which is irreducible with generic point $(t) \subset \mathbb{C}[z,t]/(t^n)$ of length n. Alternatively, we could have checked the formula $\nu_{X_n} = n$ through Equation (2.2.3) as follows. We can blow up X_n inside $M = \mathbb{A}^1 \times \mathbb{A}^1$, obtaining the exceptional divisor

$$\begin{aligned}
P(C_{X_n/\mathbb{A}^1} \oplus \mathbb{1}) &= \mathbb{P}\left[\bigoplus_{d\geq 0}(I,z)^d/(I,z)^{d+1}\right] \\
&= \mathbb{P}\left[\bigoplus_{d\geq 0}\left(R_n \cdot z^d \oplus \frac{I}{I^2}\cdot z^{d-1} \oplus \cdots \oplus \frac{I^{d-1}}{I^d}\cdot z \oplus \frac{I^d}{I^{d+1}}\right)\right] \\
&\cong \mathbb{P}\left[R_n \oplus (R_n \cdot z \oplus R_n \cdot u) \oplus (R_n \cdot z^2 \oplus R_n \cdot zu \oplus R_n \cdot u^2) \oplus \cdots\right] \\
&= \mathbb{P}R_n[z,u] = \mathbb{P}^1 \times X_n,
\end{aligned}$$

which again is irreducible with generic point of length n.

The following is an instance of both Example 2.2.3 and Example 2.2.4.

Example 2.2.6. Set $X = \operatorname{Spec} R \subset \mathbb{A}^N$, where $R = \mathbb{C}[x_1,\ldots,x_N]/(x_1^{e_1},\ldots,x_N^{e_N})$. Then, X is the critical locus of the function $\mathbb{A}^N \to \mathbb{A}^1$ sending

$$(x_1,\ldots,x_N) \mapsto \sum_{1\leq i\leq N} \frac{1}{e_i+1} x_i^{e_i+1}.$$

Thus by Example 2.2.3 we have

$$\nu_X = \mathrm{length}(X) = \prod_{1\leq i\leq N} e_i.$$

Alternatively, this formula also follows from the multiplicativity of the Behrend function, proved in general in [5, Prop. 1.5 (ii)].

So far we have only seen instances where the Behrend number of a fat point agrees with its length. In general, the length is neither an upper bound nor a lower bound for the Behrend number, as we shall see in greater detail by means of the core calculations of this chapter (see e.g. Theorem 2.3.11, Theorem 2.3.13 and Remark 2.7.9 for a few instances of this fact). For now, we present an example of a fat point with embedding dimension $N = 2$, that is neither a critical locus nor a local complete intersection.

Example 2.2.7 (Power of maximal ideal)**.** Fix an integer $d > 1$. Set $X = \operatorname{Spec} R$, where $R = \mathbb{C}[x,y]/\mathfrak{m}^d$. Here $\mathfrak{m} = (x,y)$ denotes, as ever, the maximal ideal of the origin in $\mathbb{A}^2$. We have a commutative diagram

$$\begin{tikzcd} \mathrm{Bl}_{\mathfrak{m}}\mathbb{A}^2 \arrow[r, hook] \arrow[d, "g"] & \mathbb{A}^2\times\mathbb{P}^1 \arrow[d, hook, "\mathrm{id}\times \mathsf{v}_{1,d}"] \\ \mathrm{Bl}_{\mathfrak{m}^d}\mathbb{A}^2 \arrow[r, hook] & \mathbb{A}^2\times\mathbb{P}^d \end{tikzcd} \tag{2.2.4}$$

where $\mathsf{v}_{1,d}\colon \mathbb{P}^1 \hookrightarrow \mathbb{P}^d$ is the Veronese embedding, sending $\mathbb{P}^1$ onto the rational normal curve of degree d inside $\mathbb{P}^d$. The vertical map g is an isomorphism, which by the commutativity of the diagram commutes with the projections down to $\mathbb{A}^2$. It follows that, under this isomorphism, the exceptional divisor $E \subset \mathrm{Bl}_{\mathfrak{m}^d}\mathbb{A}^2$ corresponds to the preimage of X along $\varepsilon_{\mathfrak{m}}\colon \mathrm{Bl}_{\mathfrak{m}}\mathbb{A}^2 \to \mathbb{A}^2$.

Now, as in Example 0.10.9, we can write

$$\mathrm{Bl}_{\mathfrak{m}}\mathbb{A}^2 = \left\{ ((x,y),[u:v]) \,\middle|\, xv = yu \right\} \subset \mathbb{A}^2\times\mathbb{P}^1,$$

and, after fixing coordinates (u,y) in the chart $\{v\neq 0\}\subset \mathrm{Bl}_{\mathfrak{m}}\mathbb{A}^2$, the blowp map $\varepsilon_{\mathfrak{m}}$ becomes $(u,y)\mapsto (yu,y)$ in this chart. Therefore the pullback of $X = V(\mathfrak{m}^d)$ along $\varepsilon_{\mathfrak{m}}|_{v\neq 0}$ is the scheme cut out by the ideal

$$J_v = (y^d u^d, (y^{d-1}u^{d-1})y, \ldots, (yu)y^{d-1}, y^d) = (y^d) \subset \mathbb{C}[u,y].$$

An identical calculation can be done in the chart $u\neq 0$, where one finds the ideal $J_u = (x^d)$ in $\mathbb{C}[v,x]$. All in all, $\varepsilon_{\mathfrak{m}}^{-1}(X)\subset \mathrm{Bl}_{\mathfrak{m}}\mathbb{A}^2$ (which is isomorphic to $E = E_{\mathfrak{m}^d}\mathbb{A}^2$) is defined by the ideal sheaf $\mathscr{I}^d$, where $\mathscr{I}$ is the ideal defining the (reduced) exceptional divisor in $\mathrm{Bl}_{\mathfrak{m}}\mathbb{A}^2$. It is thus a line with multiplicity d. Hence, by Lemma 2.2.2,

$$\nu_X = \mathrm{length}_{\mathscr{O}_{E,E}}\left(\mathscr{O}_{E,E}\right) = d.$$

Note that, in this case, we have $d = \nu_X < \mathrm{length}(X) = (d+1)d/2$.

The previous example can be generalised as follows.

Proposition 2.2.8. *Let $I \subset A = \mathbb{C}[x_1, \ldots, x_N]$ be an ideal of finite colength. Then, for any integer $d > 0$, one has a canonical $\mathbb{A}^N$-isomorphism $\mathrm{Bl}_{I^d} \mathbb{A}^N \cong \mathrm{Bl}_I \mathbb{A}^N$, and an identity*

$$\nu_{A/I^d} = d \cdot \nu_{A/I}.$$

In particular, if I defines a local complete intersection subscheme of $\mathbb{A}^N$, then

$$\nu_{A/I^d} = d \cdot \ell_{A/I} = d \cdot \dim_{\mathbb{C}}(A/I).$$

Proof. The second identity follows from the first combined with Example 2.2.4. We have an isomorphism of $\mathbb{A}^N$-schemes $g\colon \mathrm{Bl}_I \mathbb{A}^N \xrightarrow{\sim} \mathrm{Bl}_{I^d} \mathbb{A}^N$, which is part of a larger diagram (constructed along the same lines as Diagram 2.2.4): if we assume I is minimally generated by polynomials $f_0, f_1, \ldots, f_r \in A$, then we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Bl}_I \mathbb{A}^N \arrow[r, hook] \arrow[d, "g"] & \mathbb{A}^N \times \mathbb{P}^r \arrow[d, hook, "\mathrm{id} \times \mathsf{v}_{r,d}"] \\
\mathrm{Bl}_{I^d} \mathbb{A}^N \arrow[r, hook] & \mathbb{A}^N \times \mathbb{P}^{\binom{r+d}{d}-1}
\end{tikzcd}$$

where $\mathsf{v}_{r,d}\colon \mathbb{P}^r \to \mathbb{P}^{\binom{r+d}{d}-1}$ is the Veronese embedding.

Now, if $\varepsilon_d\colon \mathrm{Bl}_{I^d} \mathbb{A}^N \to \mathbb{A}^N$ denotes the blowup morphism, $E_d = E_{I^d}\mathbb{A}^N \subset \mathrm{Bl}_{I^d} \mathbb{A}^N$ is the exceptional divisor embedded with ideal sheaf $\mathscr{I}_d \subset \mathscr{O}_{\mathrm{Bl}_{I^d} \mathbb{A}^N}$, and the inclusion $\operatorname{Spec} A/I^d \subset \mathbb{A}^N$ has normal cone C_d, we compute

$$\begin{aligned}
\nu_{A/I^d} &= \sum_{D \subset C_d} \mathrm{mult}_{P(D)}(E_d) \\
&= \sum_{D \subset C_d} \mathrm{mult}_{P(D)}\big(V(\varepsilon_d^{-1}(I^d) \cdot \mathscr{O}_{\mathrm{Bl}_{I^d} \mathbb{A}^N})\big) \\
&= \sum_{D \subset C_d} \mathrm{mult}_{g^{-1}P(D)}\big(V(\varepsilon_1^{-1}(I^d) \cdot \mathscr{O}_{\mathrm{Bl}_I \mathbb{A}^N})\big) \\
&= \sum_{D \subset C_d} \mathrm{mult}_{g^{-1}P(D)}\big(V(\mathscr{I}_1^d)\big) \\
&= d \cdot \sum_{D \subset C_d} \mathrm{mult}_{g^{-1}P(D)}(E_1) \\
&= d \cdot \sum_{D \subset C_1} \mathrm{mult}_{P(D)}(E_1) \\
&= d \cdot \nu_{A/I},
\end{aligned}$$

as required. □

2.3 Towers and their Behrend functions

2.3.1 Towers and their basic properties

Before introducing *towers*, special ideals in $\mathbb{C}[x,y]$ particularly suited for our calculations, we quickly review some basics on curvilinear schemes.

We focus here on curvilinear schemes (cf. Definition 0.6.1) of length n supported at the origin $0 \in \mathbb{A}^2$. These are defined by ideals $I \subset \mathbb{C}[x,y]$ of the form

$$I = (f) + \mathfrak{m}^n,$$

where $f \in \mathfrak{m} \smallsetminus \mathfrak{m}^2$. Given such a polynomial $f = ax + by + cx^2 + dxy + ey^2 + \cdots$, an explicit isomorphism $\mathbb{C}[t]/t^n \to \mathbb{C}[x,y]/I$ is given by sending $t + (t^n) \mapsto (ay - bx) + I$. Such association is an isomorphism because $(a,b) \neq (0,0)$ which follows from the condition $f \in \mathfrak{m} \smallsetminus \mathfrak{m}^2$.

Definition 2.3.1. Let $f \in \mathbb{C}[x,y]$ be any nonzero polynomial, and, for $i \geq 0$, let f_i be its homogeneous part of degree i. We will denote by $o(f)$ the *order* of f, i.e.

$$o(f) = \min\left\{\, i \in \mathbb{N} \,\middle|\, f_i \neq 0 \,\right\}.$$

Lemma 2.3.2 ([8, Prop. IV.1.1])**.** *Let $I = (f) + \mathfrak{m}^n$ be a curvilinear ideal. Then, the polynomial f can be chosen in one of the following forms: either*

$$f(x,y) = x + g_x(y),$$

where $g_x \in \mathbb{C}[y]$ is such that $o(g_x) \geq 1$ and $\deg(g_x) < n$*, or*

$$f(x,y) = y + g_y(x),$$

where $g_y \in \mathbb{C}[x]$ is such that $o(g_y) \geq 1$ and $\deg(g_y) < n$.

Proposition 2.3.3. *Let $I = (f) + \mathfrak{m}^n \subset \mathbb{C}[x,y]$ be a curvilinear ideal with $n \geq 2$. Then, the blowup* $\mathrm{Bl}_I \mathbb{A}^2$ *has a Kleinian singular point of type A_{n-1} (see Section 1.1).*

Proof. By Lemma 2.3.2, we can suppose $f = x + g_x(y)$ and, as a consequence, $I = (x + g_x(y), y^n)$. In particular, the sequence $x + g_x(y), y^n$ is a regular sequence. Thus, by [21, Prop. IV-25], we have

$$\mathrm{Bl}_I \mathbb{A}^2 = \left\{ ((x,y),[u:v]) \in \mathbb{A}^2 \times \mathbb{P}^1 \,\middle|\, uf - vy^n = 0 \right\}.$$

Now, an easy computation shows that the point $((0,0),[0:1])$ is a Kleinian singularity of type A_{n-1}. □

The next definition introduces our main objects of study for this section.

Definition 2.3.4. We will say that an ideal $K \subset \mathbb{C}[x,y]$ is a *tower of height i_s* if there exists a polynomial $g_x \in \mathbb{C}[y]$ or $g_y \in \mathbb{C}[x]$, of degree strictly smaller than i_s, and a strictly increasing sequence of natural numbers $1 \leq i_1 < i_2 < \cdots < i_s$, such that

$$K = \prod_{k=1}^{s} (x + g_x(y)) + \mathfrak{m}^{i_k}$$

or

$$K = \prod_{k=1}^{s} (y + g_y(x)) + \mathfrak{m}^{i_k}.$$

We will say that a tower is

(i) *complete* if $i_k = k$ for $k = 1, \ldots, s$,

(ii) *monomial* if $g_x = 0$, in the first case, or if $g_y = 0$, in the second case.

Example 2.3.5. Let $1 \leq i_1 < i_2 < \cdots < i_s$ be a strictly increasing sequence of positive integers, and let

$$K = \prod_{k=1}^{s} (x) + \mathfrak{m}^{i_k} = \prod_{k=1}^{s} (x, y^{i_k})$$

be a monomial tower of height i_s, not necessarily complete. Then

$$K = \left(x^s, x^{s-1} y^{i_1}, x^{s-2} y^{i_1+i_2}, \ldots, x y^{\sum_{j=1}^{s-1} i_j}, y^{\sum_{j=1}^{s} i_j} \right). \tag{2.3.1}$$

The associated Ferrers diagram is depicted in Figure 2.1 in the complete case.

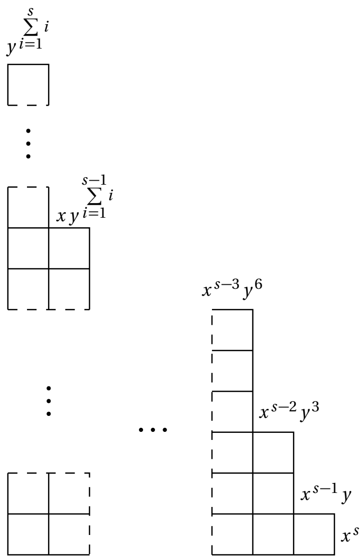


Figure 2.1. The Ferrers diagram of a complete monomial tower of height s.

Lemma 2.3.6. *The blowup of $\mathbb{A}^2$ with centre an arbitrary tower $K \subset \mathbb{C}[x, y]$ is a normal surface. Equivalently, every tower is a normal ideal.*

Proof. We first observe that the isomorphism class of a subscheme $X \subset \mathbb{A}^2$ defined by a tower is completely determined by the sequence of positive integers $1 \leq i_1 < \cdots < i_s$. Indeed, if $K = \prod_{1 \leq k \leq s} (x + g(y)) + \mathfrak{m}^{i_k}$, then the automorphism

$$\begin{array}{ccc} \mathbb{A}^2 & \xrightarrow{\sim} & \mathbb{A}^2 \\ (x, y) & \longmapsto & (x - g(y), y) \end{array} \tag{2.3.2}$$

induces an isomorphism between $V(K) \hookrightarrow \mathbb{A}^2$ and $V(K') \hookrightarrow \mathbb{A}^2$, where K' is the monomial tower $\prod_{1\le k\le s}(x)+\mathfrak{m}^{i_k}$. On the other hand, the blowup of $\mathbb{A}^2$ with centre a monomial tower $I \subset \mathbb{C}[x,y]$ is normal, because the subsets $A_I \subset \mathbb{N}^2$ and $Q_I \subset \mathbb{Q}^2$, defined as in Remark 0.10.3 and in Proposition 0.10.2 respectively, satisfy $A_I = Q_I \cap \mathbb{N}^2$. The conclusion then follows from Proposition 0.10.2. □

2.3.2 Blowing up along towers

This subsection contains the key structural results that we will need for the calculation of the Behrend function of a fat point $X \subset \mathbb{A}^2$ cut out by a tower.

Proposition 2.3.7. *Let $s \ge 1$ be a positive integer and let K_s be the complete monomial tower*

$$K_s = \prod_{k=1}^{s}(x, y^k).$$

Then, the blowup $\mathrm{Bl}_{K_s}\mathbb{A}^2$ factors as a sequence of blowups

$$X_s \xrightarrow{\varepsilon_s} X_{s-1} \xrightarrow{\varepsilon_{s-1}} \cdots \xrightarrow{\varepsilon_3} X_2 \xrightarrow{\varepsilon_2} X_1 \xrightarrow{\varepsilon_1} \mathbb{A}^2$$

where

$$\begin{aligned} X_1 &= \mathrm{Bl}_0\mathbb{A}^2, \\ X_{k+1} &= \mathrm{Bl}_{t_k} X_k, \quad k = 1,\dots,s-1. \end{aligned}$$

Here, t_1 is the toric point of $\mathrm{Exc}(\varepsilon_1) \subset X_1$ corresponding to the line $\{x=0\}$ and, for $k = 2,\dots,s-1$, t_k is the only toric point of $\mathrm{Exc}(\varepsilon_k) \smallsetminus \varepsilon_k^{-1}(\mathrm{Exc}(\varepsilon_{k-1}))$.

In other words, X_s and $\mathrm{Bl}_{K_s}\mathbb{A}^2$ are canonically isomorphic as $\mathbb{A}^2$-schemes.

Proof. For the sake of readability, we set $I_k = (x, y^k)$. The proof goes by induction on the height s of the tower. The first nontrivial case is $s = 2$. We want to prove that there exists a canonical isomorphism of $\mathbb{A}^2$-schemes $\varphi\colon X_2 \to \mathrm{Bl}_{K_2}\mathbb{A}^2$.

Lemma 0.2.1 implies that

$$\mathrm{Bl}_{K_2}\mathbb{A}^2 \cong \mathrm{Bl}_{\varepsilon_1^{-1}(I_2)\cdot\mathscr{O}_{X_1}} X_1.$$

Recall (see [13, § 3.1]) that X_1 is a toric surface covered by two toric charts $U_i \cong \mathbb{A}^2$, for $i = 0,1$, with the property that, if we call a_i, b_i the toric coordinates on U_i, then the maps $\varepsilon_1|_{U_i}$, for $i = 0,1$, take the the form

$$\begin{aligned} \varepsilon_1\big|_{U_0}(a_0,b_0) &= (a_0b_0, b_0) \\ \varepsilon_1\big|_{U_1}(a_1,b_1) &= (a_1, a_1b_1). \end{aligned}$$

As a consequence

$$\begin{aligned} \varepsilon_1\big|_{U_0}^{-1}(I_2)\cdot\mathbb{C}[a_0,b_0] &= (a_0b_0, b_0^2) = (b_0)\cdot(a_0,b_0), \\ \varepsilon_1\big|_{U_1}^{-1}(I_2)\cdot\mathbb{C}[a_1,b_1] &= (a_1, a_1^2b_1^2) = (a_1). \end{aligned}$$

Therefore, we conclude that $\varepsilon_1^{-1}(I_2)\cdot\mathscr{O}_{X_1}=\mathscr{H}_1\cdot\mathscr{H}_2$ where $\mathscr{H}_1\subset\mathscr{O}_{X_1}$ defines a Cartier divisor and $\mathscr{H}_2\subset\mathscr{O}_{X_1}$ defines a (reduced) toric point $t_1\in X_1$. Thus, we have

$$\mathrm{Bl}_{\mathscr{H}_1\cdot\mathscr{H}_2}X_1\cong\mathrm{Bl}_{t_1}X_1=X_2,$$

which concludes the proof of the base step.

Suppose now that we have a canonical isomorphism of $\mathbb{A}^2$-schemes $\varphi_s\colon X_s\xrightarrow{\sim}\mathrm{Bl}_{K_s}\mathbb{A}^2$. We need to construct a canonical isomorphism

$$\varphi_{s+1}\colon X_{s+1}\xrightarrow{\ \sim\ }\mathrm{Bl}_{K_{s+1}}\mathbb{A}^2.$$

Setting $\psi_s=\varepsilon_1\circ\cdots\circ\varepsilon_s$, we have a commutative diagram

$$\begin{tikzcd}
X_{s+1} \arrow[r, "\overline{\varphi}_s"] \arrow[d, "\varepsilon_{s+1}"'] \arrow[rr, bend left, dotted, "\varphi_{s+1}"] & \mathrm{Bl}_{\varphi_s(t_s)}\mathrm{Bl}_{K_s}\mathbb{A}^2 \arrow[d, "\varepsilon_{\varphi_s(t_s)}"] & \mathrm{Bl}_{K_{s+1}}\mathbb{A}^2 \arrow[d, "\varepsilon_{K_{s+1}}"] \\
X_s \arrow[r, "\varphi_s"] \arrow[rr, bend right, "\psi_s"'] & \mathrm{Bl}_{K_s}\mathbb{A}^2 \arrow[r, "\varepsilon_{K_s}"] & \mathbb{A}^2
\end{tikzcd}$$

where the map $\overline{\varphi}_s$ is an isomorphism by the base change properties of blowups [21, Prop. IV-21].

Now, we exploit the toric variety structure on X_i, for all $i\in\mathbb{N}$. If N denotes the standard 2-dimensional lattice, then the variety X_s can be constructed via the fan Σ_s in $N\otimes_{\mathbb{Z}}\mathbb{R}\cong\mathbb{R}^2$ depicted in Figure 2.2.

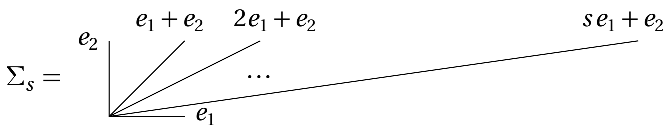


Figure 2.2. A fan realising the toric variety X_s.

The variety X_s is covered by $s+1$ smooth charts $U_k=\operatorname{Spec}S_k\cong\mathbb{A}^2$ (see Section 1.1) with toric coordinates a_k and b_k, more precisely we set

$$\begin{aligned}
S_k&=\mathbb{C}[xy^{-k+1},x^{-1}y^k]=\mathbb{C}[a_k,b_k], \qquad 1\le k\le s\\
S_{s+1}&=\mathbb{C}[xy^{-s},y]=\mathbb{C}[a_{s+1},b_{s+1}].
\end{aligned}$$

As above, the maps $\psi_s|_{U_i}\colon U_i\to\mathbb{A}^2$, for $i=1,\dots,s+1$, have the explicit description

$$\begin{aligned}
\psi_s\big|_{U_k}(a_k,b_k)&=(a_k^kb_k^{k-1},a_kb_k), \qquad 1\le k\le s\\
\psi_s\big|_{U_{s+1}}(a_{s+1},b_{s+1})&=(a_{s+1}b_{s+1}^s,b_{s+1}).
\end{aligned}$$

Therefore, the ideal sheaf $\psi_s^{-1}(I_{s+1})\cdot\mathscr{O}_{X_s}\subset\mathscr{O}_{X_s}$ is given, locally on each chart, by

$$\begin{aligned}
\psi_s\big|_{U_k}^{-1}(I_{s+1})\cdot\mathbb{C}[a_k,b_k]&=(a_k^kb_k^{k-1},a_k^{s+1}b_k^{s+1})=(a_k^kb_k^{k-1})\subset S_k, \qquad 1\le k\le s\\
\psi_s\big|_{U_{s+1}}^{-1}(I_{s+1})\cdot\mathbb{C}[a_{s+1},b_{s+1}]&=(a_{s+1}b_{s+1}^s,b_{s+1}^{s+1})=(b_{s+1}^s)\cdot(a_{s+1},b_{s+1})\subset S_{s+1}.
\end{aligned}$$

As a consequence, $\psi_s^{-1}(I_{s+1})\cdot\mathscr{O}_{X_s}=\mathscr{H}_1\cdot\mathscr{H}_2$ where, as above, $\mathscr{H}_1\subset\mathscr{O}_{X_s}$ defines a Cartier divisor on X_s and $\mathscr{H}_2$ defines the reduced toric point $t_s\in\operatorname{Exc}(\varepsilon_s)\setminus\varepsilon_s^{-1}(\operatorname{Exc}(\varepsilon_{s-1}))$. Finally, the statement follows by applying again Lemma 0.2.1. □

Corollary 2.3.8. *Let K be the complete tower $K=\prod_{1\le k\le s}(x+g(y))+\mathfrak{m}^k$, where $g(y)\in\mathbb{C}[y]$ is a polynomial of order $o(g)\ge 1$ and degree $\deg(g)<s$. Then, the blowup $\varepsilon_K\colon\operatorname{Bl}_K\mathbb{A}^2\to\mathbb{A}^2$ factors as a sequence of blowups*

$$\operatorname{Bl}_{p_{s-1}}\operatorname{Bl}_{p_{s-2}}\cdots\operatorname{Bl}_{p_1}\operatorname{Bl}_0\mathbb{A}^2\xrightarrow{\varepsilon_s}\cdots\xrightarrow{\varepsilon_3}\operatorname{Bl}_{p_1}\operatorname{Bl}_0\mathbb{A}^2\xrightarrow{\varepsilon_2}\operatorname{Bl}_0\mathbb{A}^2\xrightarrow{\varepsilon_1}\mathbb{A}^2$$

where $p_1\in\operatorname{Exc}(\varepsilon_1)$ and, for all $k=2,\dots,s-1$, the point p_k belongs to $\operatorname{Exc}(\varepsilon_k)\setminus\varepsilon_k^{-1}\operatorname{Exc}(\varepsilon_{k-1})$.

Proof. It is enough to combine Proposition 2.3.7 with the automorphism (2.3.2) introduced in the proof of Lemma 2.3.6. □

Remark 2.3.9. The above corollary, combined with Lemma 2.3.6, also allows one to handle the blowup of $\mathbb{A}^2$ along *any* tower

$$K=\prod_{k=1}^{s}(x+g(y))+\mathfrak{m}^{i_k}.$$

Indeed, given the complete tower $\overline{K}$ defined by

$$\overline{K}=\prod_{k=1}^{i_s}(x+g(y))+\mathfrak{m}^{k},$$

the blowup $B=\operatorname{Bl}_K\mathbb{A}^2$ can be obtained by contracting some projective lines in $\overline{B}=\operatorname{Bl}_{\overline{K}}\mathbb{A}^2$.

In a little more detail, if we call $\varepsilon\colon B\to\mathbb{A}^2$ and $\bar\varepsilon\colon\overline{B}\to\mathbb{A}^2$ the blowup maps, the same computations as in the proof of Proposition 2.3.7 show that $\bar\varepsilon^{-1}(K)\cdot\mathscr{O}_{\overline{B}}$ defines a Cartier divisor on $\overline{B}$. Therefore, there is a canonical birational morphism of $\mathbb{A}^2$-schemes

$$\varphi\colon\overline{B}\to B$$

which has connected fibres by Zariski's Main Theorem (Theorem 0.9.2) (which we may apply since B is normal, by Lemma 2.3.6). In particular, the map φ is an isomorphism outside from the respective exceptional loci of $\overline{B}$ and B and it may contract some of the irreducible components of $\operatorname{Exc}(\overline{B})$.

Since any tower is isomorphic to a monomial tower (see the proof of Lemma 2.3.6), in order to understand which rational projective curves of $\overline{B}$ are contracted by φ, we can first suppose that K is a monomial tower. Then, the usual toric geometry methods apply. A fan Σ for the toric variety B consists of the following $s+1$ maximal cones

$$\begin{aligned}\sigma_0&=\langle e_2,i_1e_1+e_2\rangle,\\ \sigma_1&=\langle i_1e_1+e_2,i_2e_1+e_2\rangle,\\ &\ \vdots\\ \sigma_{s-1}&=\langle i_{s-1}e_1+e_2,i_se_1+e_2\rangle,\\ \sigma_s&=\langle i_se_1+e_2,e_1\rangle.\end{aligned}$$

In particular, if we put $i_0 = 0$, the cones σ_j, for $j = 0, \dots, s-1$, correspond either to a smooth point, if $i_{j+1} - i_j - 1 = 0$, or to a Kleininian singularity of type $A_{i_{j+1}-i_j-1}$ otherwise, whereas the cone σ_s corresponds to a smooth point of B. Now, a fan $\overline{\Sigma}$ for the toric variety $\overline{B}$ has the following maximal cones

$$\begin{aligned}
\tau_0 &= \langle e_2, e_1 + e_2 \rangle, \\
\tau_1 &= \langle e_1 + e_2, 2e_1 + e_2 \rangle, \\
&\vdots \\
\tau_{i_s - 1} &= \langle (i_s - 1)e_1 + e_2, i_s e_1 + e_2 \rangle, \\
\tau_{i_s} &= \langle i_s e_1 + e_2, e_1 \rangle.
\end{aligned}$$

Moreover, the fact that each cone τ_i of $\overline{\Sigma}$ is contained in some cone σ_j of Σ implies that there is a morphism of $\mathbb{A}^2$-schemes from $\overline{B}$ to B which, by universality, must coincide with φ. Therefore, the lines contracted by φ are the lines in $\overline{B}$ corresponding to the rays of $\overline{\Sigma}$ not belonging to Σ.

Notice also that, if one knows how to compute the Behrend number of a complete tower (which we do, as we shall see in Theorem 2.3.11), then, thanks to this remark, one also knows how to compute the Behrend number of an arbitrary tower. To see this, consider a curve $C \subset \overline{B}$ which is not contracted by φ. Then, since, when restricted to the complement $U \subset \overline{B}$ of the contracted lines, φ is an isomorphism, we have the identity

$$\operatorname{mult}_{\varphi(C)}(V(\varepsilon^{-1}(K) \cdot \mathscr{O}_B)) = \operatorname{mult}_C(V(\bar{\varepsilon}^{-1}(K) \cdot \mathscr{O}_{\overline{B}})).$$

The following example provides a generalisation of [21, Prop. IV-40].

Example 2.3.10. The easiest non-complete tower one can think of is given by a curvilinear ideal $I = (x) + \mathfrak{m}^n = (x, y^n)$ with $n \geq 2$. We can deduce, from the above remark, an alternative way to Proposition 2.3.3, to prove that $\mathrm{Bl}_I \mathbb{A}^2$ has an (isolated) singularity of type A_{n-1}.

Let K be the monomial complete tower

$$K = \prod_{k=1}^{n} (x) + \mathfrak{m}^k = \prod_{k=1}^{n} (x, y^k).$$

Then, Proposition 2.3.7 implies that the exceptional locus of the map $\varepsilon_K \colon \mathrm{Bl}_K \mathbb{A}^2 \to \mathbb{A}^2$ is a chain of n rational smooth projective curves

$$E_1 \cup E_2 \cup \cdots \cup E_n \subset \mathrm{Bl}_K \mathbb{A}^2$$

and, the classical blowup formula (see [3, I-§ 9, Thm. (9.1)]), implies:

$$E_k^2 = \begin{cases} -2 & \text{if } k = 1, \dots, n-1 \\ -1 & \text{if } k = n. \end{cases}$$

Now, as a consequence of Remark 2.3.9, the canonical projective birational morphism

$$\varphi \colon \mathrm{Bl}_K \mathbb{A}^2 \to \mathrm{Bl}_I \mathbb{A}^2$$

contracts the curves

$$E_1, \dots, E_{n-1},$$

and, the characterisation of Kleinian singularities (see [3, III-§ 3, Prop. (3.4)]), implies that $\mathrm{Bl}_I \mathbb{A}^2$ has an Kleinian singularity of type A_{n-1}.

2.3.3 The Behrend function of a tower

We are ready to tackle the calculation of the Behrend number of a tower. We start with the complete case.

Theorem 2.3.11. *Let $K_s \subset \mathbb{C}[x,y]$ be a complete tower of height s. Then*

$$\ell_{\mathbb{C}[x,y]/K_s} = \binom{s+2}{3}, \tag{2.3.3}$$

$$\nu_{\mathbb{C}[x,y]/K_s} = \frac{s(s+1)(2s+1)}{6}. \tag{2.3.4}$$

In particular $\ell_{\mathbb{C}[x,y]/K_s} < \nu_{\mathbb{C}[x,y]/K_s}$ for all $s > 1$.

Proof. Equation (2.3.3) follows directly from Example 2.3.5 together with the equality[1]

$$\binom{s+2}{3} = \frac{s(s+1)(s+2)}{6} = \sum_{k=1}^{s}\sum_{i=1}^{k} i = \sum_{j=0}^{s-1}\sum_{i=1}^{j+1} i.$$

We now prove Equation (2.3.4). Let $D = \psi_s^{-1}(V(K_s))$ be the subscheme of $X_s = \mathrm{Bl}_{t_{s-1}} \cdots \mathrm{Bl}_{t_1} \mathrm{Bl}_0 \mathbb{A}^2$, where the points t_i are as in Proposition 2.3.7, defined as the scheme-theoretic preimage of $V(K_s) \subset \mathbb{A}^2$ via the iterated blowup map $\psi_s \colon X_s \to \mathbb{A}^2$. Then, Proposition 2.3.7 allows us to identify the $\mathbb{A}^2$-schemes X_s and $\mathrm{Bl}_{K_s} \mathbb{A}^2$ and, as a consequence, to compute the Behrend number of the ideal K_s as

$$\nu_{\mathbb{C}[x,y]/K_s} = \sum_{C \subset D} \mathrm{mult}_C D,$$

where the sum ranges over all irreducible components C of D. Notice that, if $\varepsilon \colon \mathrm{Bl}_{K_s} \mathbb{A}^2 \to \mathbb{A}^2$ denotes the blowup morphism then, under the canonical isomorphism $X_s \cong \mathrm{Bl}_{K_s} \mathbb{A}^2$, the exceptional locus $\mathrm{Exc}(\varepsilon)$ corresponds to D_{red}.

Recall that D_{red} is a chain of smooth rational projective curves $C_1, \dots, C_s$, where C_1 corresponds to the blowup of the origin of $\mathbb{A}^2$ under the isomorphism of Proposition 2.3.7 and

$$C_i \cap C_j = \begin{cases} \text{one point} & \text{if } |i-j| = 1 \\ \emptyset & \text{if } |i-j| > 1. \end{cases}$$

We thus have to compute the sum

$$\sum_{i=1}^{s} \mathrm{mult}_{C_i} D.$$

Recall (see Section 1.1) also that X_s is covered by $s+1$ charts isomorphic to $\mathbb{A}^2$ defined by

$$U_k = \operatorname{Spec} \mathbb{C}[xy^{-k+1}, x^{-1}y^k], \qquad 1 \le k \le s$$
$$U_{s+1} = \operatorname{Spec} \mathbb{C}[xy^{-s}, y].$$

[1] Such number is known as the s-th *tetrahedral number*.

If a_k, b_k are the toric coordinates on U_k for $k = 1, \dots, s+1$, the map ψ_s is

$$\begin{aligned} \psi_s\big|_{U_k}(a_k, b_k) &= (a_k^k b_k^{k-1}, a_k b_k), \qquad 1 \le k \le s \\ \psi_s\big|_{U_{s+1}}(a_{s+1}, b_{s+1}) &= (a_{s+1} b_{s+1}^s, b_{s+1}). \end{aligned}$$

Therefore, the ideal sheaf $\psi_s^{-1}(K_s) \cdot \mathscr{O}_{X_s} \subset \mathscr{O}_{X_s}$ is given, locally on each chart, by

$$\begin{aligned} \psi_s\big|_{U_k}^{-1}(K_s)\mathbb{C}[a_k, b_k] &= (a_k^k b_k^{k-1}, a_k b_k)(a_k^k b_k^{k-1}, a_k^2 b_k^2) \cdots (a_k^k b_k^{k-1}, a_k^k b_k^k) \cdots (a_k^k b_k^{k-1}, a_k^s b_k^s) \\ &= (a_k b_k \cdot a_k^2 b_k^2 \cdots a_k^{k-1} b_k^{k-1} \cdot a_k^k b_k^{k-1} \cdots a_k^k b_k^{k-1}) \qquad \text{for } k \le s \\ \psi_s\big|_{U_{s+1}}^{-1}(K_s)\mathbb{C}[a_{s+1}, b_{s+1}] &= (a_{s+1} b_{s+1}^s, b_{s+1})(a_{s+1} b_{s+1}^s, b_{s+1}^2) \cdots (a_{s+1} b_{s+1}^s, b_{s+1}^s) \\ &= (b_{s+1} b_{s+1}^2 \cdots b_{s+1}^s). \end{aligned}$$

We can read from the above formulas the contribution a_{ij} of the curvilinear ideal $(x) + \mathfrak{m}^i$ to the multiplicity of the component C_j of the exceptional divisor of ε. This information is encoded in the matrix

$$A = (a_{i,j})_{i,j \in \{1,\dots,s\}} = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 2 & \cdots & 2 & 2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 2 & \cdots & s-1 & s-1 \\ 1 & 2 & \cdots & s-1 & s \end{pmatrix}.$$

For instance, the last column is given by the vector of the exponents of b_{s+1} in the last displayed equation. Notice that, $a_{ij} = \min\{i, j\}$.

The Behrend number of K_s is

$$\nu_{\mathbb{C}[x,y]/K_s} = \sum_{i,j \in \{1,\dots,s\}} a_{ij}.$$

In order to complete the proof we observe that

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & 1 & 1 \\ 0 & 1 & \cdots & 1 & 1 \end{pmatrix} + \cdots + \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Hence, we have

$$\nu_{\mathbb{C}[x,y]/K_s} = \sum_{k=1}^{s} k^2 = \frac{s(s+1)(2s+1)}{6},$$

which completes the proof. □

Comparing Behrend functions, we obtain the following easy corollary.

Corollary 2.3.12. *A complete tower of height at least* 2 *is not a curvilinear ideal, i.e. it has embedding dimension* 2.

Following the prescriptions in Remark 2.3.9, with similar techniques, one can prove the following generalisation of Theorem 2.3.11.

Theorem 2.3.13. *Let $1 \le i_1 < \cdots < i_s$ be a strictly increasing sequence of positive integers and let $K = \prod_{1 \le k \le s}(x + f(y)) + \mathfrak{m}^{i_k}$ be a tower of height i_s. Then there are identities*

$$\ell_{\mathbb{C}[x,y]/K} = \sum_{k=1}^{s}\sum_{j=1}^{k} i_j,$$

$$\nu_{\mathbb{C}[x,y]/K} = \ell_{\mathbb{C}[x,y]/K} + \sum_{j=1}^{s-1} i_j(s-j).$$

We have thus computed the length and the Behrend number of an arbitrary tower, and the latter happens to be greater than the former.

2.3.4 Products of towers, Dynkin diagrams and Behrend functions

Definition 2.3.14. Let X be a smooth quasiprojective surface and let $C_1, \ldots, C_s \subset X$ be s distinct rational smooth projective curves with the property that $C_i \cap C_j$ is either empty or a singleton for $i \neq j$. We will call *Dynkin diagram* of the set of curves $\{ C_i \mid i = 1, \ldots, s \}$ a diagram made of:

(i) s circles that we will call nodes, each labeled by one of the curves, and decorated with its self-intersection,

(ii) for any $i \neq j$ such that $C_i \cap C_j \neq \emptyset$, a segment called edge joining the nodes labeled by C_i and C_j.

Example 2.3.15. Let $K = \prod_{1 \le i \le s} I_i$ be a complete tower. Then, as explained in Corollary 2.3.8 the variety $X = \mathrm{Bl}_K \mathbb{A}^2$ can be obtained after a sequence of blowups each with centre a reduced point and hence, X is smooth. Moreover, the exceptional locus $\mathrm{Exc}(\varepsilon)$ of the blowup map

$$\varepsilon \colon X \to \mathbb{A}^2$$

consists of a chain of rational smooth projective curves $\mathcal{D} = \{C_1, \ldots, C_s\}$. In particular, they satisfy the same property as the curves in Definition 2.3.14.

Notice that, for all $j = 1, \ldots, s$ the ideal sheaf $\varepsilon^{-1}(I_j) \cdot \mathscr{O}_X \subset \mathscr{O}_X$ defines a Cartier divisor. As a consequence, we can associate, to each ideal I_j one of the curves C_i. We say that the curve C_i corresponds to the ideal I_j if the canonical morphism

$$\varphi \colon X \to \mathrm{Bl}_{I_j} \mathbb{A}^2$$

contracts all the curves $C_k \subset X$ for $k \neq i$ (see Example 2.3.10). Notice that this association is well defined. Indeed, $\mathrm{Bl}_{I_j} \mathbb{A}^2$ is normal by Lemma 2.3.6 and hence φ has connected fibres by Theorem 0.9.2. As a consequence, only one of the curves C_i can map bijectively onto the irreducible rational curve $\mathrm{Exc}(\mathrm{Bl}_{I_j} \mathbb{A}^2)$.

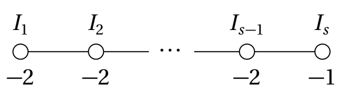


Figure 2.3. The Dynkin diagram of the tower K, with each node labeled by an ideal.

Sometimes, in the literature (see e.g. [8]), the underlying unlabeled diagram is called *bamboo.*

Lemma 2.3.16. *Let us consider the product two complete monomial towers of height h, of the form*

$$I_h = \left(\prod_{k=1}^{h}(x)+\mathfrak{m}^k\right)\cdot\left(\prod_{k=1}^{h}(y)+\mathfrak{m}^k\right).$$

Then, the numbers $\{\ell_{\mathbb{C}[x,y]/I_h} \mid h \geq 1\}$ satisfy the recursive relation

$$\ell_{\mathbb{C}[x,y]/I_h} = \ell_{\mathbb{C}[x,y]/I_{h-1}} + h^2 + 3h - 1.$$

Equivalently, we have

$$\ell_{\mathbb{C}[x,y]/I_h} = \frac{h(h+1)(h+2)}{3} + h^2,$$

which in turn equals $2\cdot\ell + h^2$, where ℓ is the colength of the tower $\prod_{k=1}^{h}(x)+\mathfrak{m}^k$.

Proof. The equivalence between the two formulas is straightforward to check and we will omit it. We now prove the former.

For each $h \geq 1$, we have

$$\begin{aligned} I_h &= \mathfrak{m}^2(x,y^2)(x^2,y)\cdots(x,y^h)(x^h,y) \\ &= (x^2,xy,y^2)(x^3,xy,y^3)\cdots(x^{h+1},xy,y^{h+1}). \end{aligned}$$

Then I_h, a product of h monomial ideals, can be generated by $2h+1$ monomials, namely we have

$$I_h = \left(x^{h-i}y^{h+\binom{i+1}{2}}, x^{h+\binom{i+1}{2}}y^{h-i} \,\middle|\, 0 \leq i \leq h\right). \tag{2.3.5}$$

A few examples are given in Figure 2.4. The integers

$$a_h = h + \binom{h+1}{2}, \qquad h \geq 1, \tag{2.3.6}$$

represent the maximal power of x (equivalently, of y) appearing among the $2h+1$ generators of I_h. The colength of I_h is computed, thanks to Equation (2.3.5), in a recursive way from the base case $\ell_{\mathbb{C}[x,y]/I_1} = 3$. We obtain

$$\ell_{\mathbb{C}[x,y]/I_h} = \ell_{\mathbb{C}[x,y]/I_{h-1}} + 2a_h - 1 = \ell_{\mathbb{C}[x,y]/I_{h-1}} + h^2 + 3h - 1.$$

as required. □

The induction described in the proof works as depicted in Figure 2.4 below.

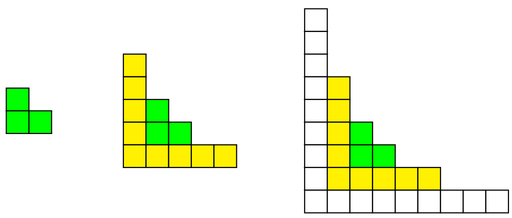


Figure 2.4. The ideals I_h for $h = 1,2,3$. The lengths are 3, 12, 29, and the heights of the respective Ferrers diagrams are $a_1 = 2$, $a_2 = 5$, $a_3 = 9$.

Theorem 2.3.17. *The following properties hold for complete towers.*

1. *Let K_x and K_y be two complete towers, of heights h_x and h_y respectively, of the form*

$$K_x = \prod_{k=1}^{h_x} (x + g_x(y)) + \mathfrak{m}^k, \qquad K_y = \prod_{k=1}^{h_y} (y + g_y(x)) + \mathfrak{m}^k,$$

for some $g_x \in \mathbb{C}[y]$ and $g_y \in \mathbb{C}[x]$ such that

$$[(x + g_x) + \mathfrak{m}^2] \neq [(y + g_y) + \mathfrak{m}^2] \in \mathbb{P}(\mathfrak{m}/\mathfrak{m}^2).$$

Then

$$\ell_{\mathbb{C}[x,y]/K_x \cdot K_y} = \ell_{\mathbb{C}[x,y]/K_x} + \ell_{\mathbb{C}[x,y]/K_y} + h_x h_y \tag{2.3.7}$$

$$\nu_{\mathbb{C}[x,y]/K_x \cdot K_y} = \nu_{\mathbb{C}[x,y]/K_x} + \nu_{\mathbb{C}[x,y]/K_y} + 2h_x h_y - h_x - h_y. \tag{2.3.8}$$

2. *Let K_1 and K_2 be two complete towers, of height respectively h_1 and h_2, of the form*

$$K_1 = \prod_{k=1}^{h_1} (x + g_1(y)) + \mathfrak{m}^k, \qquad K_2 = \prod_{k=1}^{h_2} (x + g_2(y)) + \mathfrak{m}^k.$$

for some $g_1 \neq g_2 \in \mathbb{C}[y]$ of respective degrees

$$\deg(g_1) < h_1 \textit{ and } \deg(g_2) < h_2.$$

Let $d = o(g_1 - g_2)$ be the order of $g_1 - g_2 \in \mathbb{C}[y]$. Then

$$\nu_{\mathbb{C}[x,y]/K_1 \cdot K_2} = \nu_{\mathbb{C}[x,y]/K_1} + \nu_{\mathbb{C}[x,y]/K_2} + 2h_1 h_2 - d(h_1 + h_2). \tag{2.3.9}$$

Proof. First of all, an easy computation shows that, since $[(x + g_x) + \mathfrak{m}^2] \neq [(y + g_y) + \mathfrak{m}^2]$ are different points in $\mathbb{P}(\mathfrak{m}/\mathfrak{m}^2)$, one also has $[(x - g_x) + \mathfrak{m}^2] \neq [(y - g_y) + \mathfrak{m}^2] \in \mathbb{P}(\mathfrak{m}/\mathfrak{m}^2)$. In particular, the Jacobian of the map

$$\begin{aligned} \mathbb{A}^2 &\xrightarrow{\quad\psi\quad} \mathbb{A}^2 \\ (x, y) &\longmapsto (x - g_x(y), y - g_y(x)) \end{aligned}$$

has maximal rank at the origin $0 \in \mathbb{A}^2$, i.e. it is a biholomorphism near the origin. Such observation ensures that, in order to prove (1), it is enough to prove the statement for $g_x = g_y = 0$. Therefore, we have reduced to the case

$$\begin{aligned} K_x &= \prod_{k=1}^{h_x} (x) + \mathfrak{m}^k = \prod_{k=1}^{h_x} (x, y^k) \\ K_y &= \prod_{k=1}^{h_y} (y) + \mathfrak{m}^k = \prod_{k=1}^{h_y} (x^k, y). \end{aligned}$$

The statement about the length is already proved in the case $h_x = h_y$ (Lemma 2.3.16). We prove the general case via an inductive argument. Let us assume, without loss of generality, that $e = h_x - h_y > 0$. We argue by induction on e. We set $h = h_y$ and we denote by $K_\bullet^{(t)}$, for $t \in \mathbb{N}$, the tower

$$K_\bullet^{(t)} = \prod_{k=1}^{t} (\bullet) + \mathfrak{m}^k,$$

for $\bullet \in \{x, y\}$.

Step 1. Assume $e = 1$ (so that $h_x = h+1$). To prove Equation (2.3.7) in this case, it is enough to observe that the Ferrers diagram of the ideal

$$K_x^{(h+1)} \cdot K_y^{(h)} = (x, y^{h+1}) \cdot K_x^{(h)} \cdot K_y^{(h)} = (x) \cdot K_x^{(h)} \cdot K_y^{(h)} + (y^{a_h+h+1}).$$

is obtained from the Ferrers diagram of $K_x^{(h)} \cdot K_y^{(h)}$ by shifting it to the right by one position and adding a column of height $a_h + h + 1$ to the left, where a_h is defined as in Equation (2.3.6). Thus the colength of $K_x^{(h_x)} \cdot K_y^{(h_y)} \subset \mathbb{C}[x, y]$ is

$$\begin{aligned}
\ell_{\mathbb{C}[x,y]/K_x^{(h+1)} \cdot K_y^{(h)}} &= \ell_{\mathbb{C}[x,y]/K_x^{(h)} \cdot K_y^{(h)}} + a_h + h + 1 \\
&= h^2 + 2 \cdot \binom{h+2}{3} + \binom{h+1}{2} + 2h + 1,
\end{aligned}$$

where we have exploited Lemma 2.3.16 in the last equality. It is straightforward to check that this number agrees with

$$\binom{h+3}{3} + \binom{h+2}{3} + (h+1)h = \ell_{\mathbb{C}[x,y]/K_x^{(h_x)}} + \ell_{\mathbb{C}[x,y]/K_y^{(h_y)}} + h_x h_y.$$

So the base of the induction is proved.

Step 2. Now we assume Equation (2.3.7) up to e and we prove the formula for $e+1$ (so now $h = h_y$ and $h_x = h + e + 1$). The ideal we have to compute the length of is

$$I_{h,e+1} = K_x^{(h+e+1)} \cdot K_y^{(h)} = \left(x, y^{h+1}\right)\left(x, y^{h+2}\right) \cdots \left(x, y^{h+e}\right)\left(x, y^{h+e+1}\right) \cdot K_x^{(h)} \cdot K_y^{(h)}.$$

By direct calculation, or by an application of Equation (2.3.1) taken with $s = e$ and $i_k = h + k$, one finds that, for every $e > 0$, there is an identity

$$\left(x, y^{h+1}\right)\left(x, y^{h+2}\right) \cdots \left(x, y^{h+e}\right) = \left(x^i y^{(e-i)h + \binom{e+1-i}{2}} \,\middle|\, 0 \le i \le e\right).$$

We know, by the inductive hypothesis, that

$$\begin{aligned}
\ell_{\mathbb{C}[x,y]/I_{h,e}} &= \ell_{\mathbb{C}[x,y]/K_x^{(h+e)}} + \ell_{\mathbb{C}[x,y]/K_y^{(h)}} + (h+e)h \\
&= \binom{h+e+2}{3} + \binom{h+2}{3} + (h+e)h.
\end{aligned}$$

As in **Step 1**, the Ferrers diagram of the ideal

$$I_{h,e+1} = \left(x, y^{h+e+1}\right) \cdot I_{h,e} = (x) \cdot I_{h,e} + \left(y^{h+e+1}\right)$$

is obtained from the Ferrers diagram of $I_{h,e}$ by shifting it to the right by one position, and adding a column of height

$$\binom{e+1}{2} + eh + \binom{h+1}{2} + h + (h+e+1)$$

to the left. The number $\binom{e+1}{2} + eh + \binom{h+1}{2} + h$ is the height of the Ferrers diagram of $I_{h,e}$. We obtain

$$\ell_{\mathbb{C}[x,y]/I_{h,e+1}} = \ell_{\mathbb{C}[x,y]/I_{h,e}} + \binom{e+1}{2} + eh + \binom{h+1}{2} + h + (h+e+1).$$

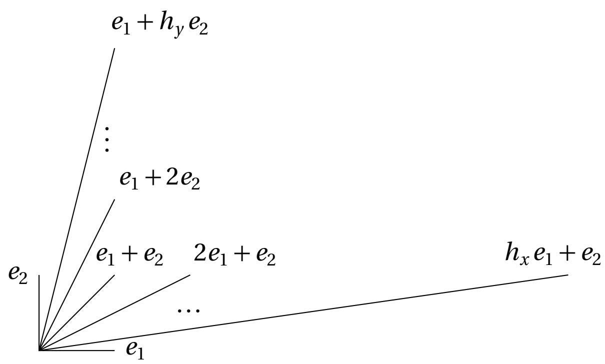
It is now straightforward to check that this number agrees with

$$\begin{aligned}\binom{h+e+3}{3} + \binom{h+2}{3} + (h+e+1)h &= \ell_{\mathbb{C}[x,y]/K_x^{(h+e+1)}} + \ell_{\mathbb{C}[x,y]/K_y^{(h)}} + (h+e+1)h \\ &= \ell_{\mathbb{C}[x,y]/K_x^{(h_x)}} + \ell_{\mathbb{C}[x,y]/K_y^{(h_y)}} + h_x \cdot h_y.\end{aligned}$$

So we have proved Equation (2.3.7).

We now move to proving Equation (2.3.8). This equation is implied by the more general Equation (2.3.9), whose proof is essentially equivalent to that of Equation (2.3.8). Therefore we will give full details on the former and precise indications on how to prove the latter.

We shall use the shorthand notation $I = K_x \cdot K_y$. Lemma 0.2.1 implies, together with the usual toric construction, that there is a canonical isomorphism of $\mathbb{A}^2$-schemes $\varphi\colon Y \to \mathrm{Bl}_I \mathbb{A}^2$ where Y is the toric variety with the following fan

$\Sigma =$ 

and the structure of $\mathbb{A}^2$-scheme of Y is induced by the identity map of the standard lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$.

Now, as in the proof of Theorem 2.3.11, we can create a table encoding the contribution of each ideal $I_{x,i} = (x) + \mathfrak{m}^i$ and $I_{y,j} = (y) + \mathfrak{m}^j$, for $i = 1, \ldots, h_x$ and $j = 1, \ldots, h_y$, to the multiplicity of each irreducible component of the exceptional divisor of the blowup $\mathrm{Bl}_I \mathbb{A}^2$. Such table has the following form.

	I_{y,h_y}	I_{y,h_y-1}	$\cdots$	$I_{y,2}$	$\mathfrak{m}$	$I_{x,2}$	$\cdots$	I_{x,h_x-1}	I_{x,h_x}
	-1	-2		-2	-3	-2		-2	-1
$I_{x,1}=\mathfrak{m}$	1	1	$\cdots$	1	1	1	$\cdots$	1	1
$I_{x,2}$	1	1	$\cdots$	1	1	2	$\cdots$	2	2
$\vdots$	$\vdots$	$\vdots$	$\iddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
I_{x,h_x-1}	1	1	$\cdots$	1	1	2	$\cdots$	h_x-1	h_x-1
I_{x,h_x}	1	1	$\cdots$	1	1	2	$\cdots$	h_x-1	h_x
$I_{y,1}=\mathfrak{m}$	1	1	$\cdots$	1	1	1	$\cdots$	1	1
$I_{y,2}$	2	2	$\cdots$	2	1	1	$\cdots$	1	1
$\vdots$	$\vdots$	$\vdots$	$\iddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
I_{y,h_y-1}	h_y-1	h_y-1	$\cdots$	2	1	1	$\cdots$	1	1
I_{y,h_y}	h_y	h_y-1	$\cdots$	2	1	1	$\cdots$	1	1

Now, the Behrend number is the sum of all entries of the above table and, the equality

$$\nu_{\mathbb{C}[x,y]/I} = \nu_{\mathbb{C}[x,y]/(K_x\cdot K_y} = \nu_{\mathbb{C}[x,y]/K_x} + \nu_{\mathbb{C}[x,y]/K_y} + 2h_x h_y - h_x - h_y$$

is obtained similarly as in the proof of Theorem 2.3.11. Finally, (1) is proved.

In order to prove (2), we reduce to the simpler case $g = g_1 = -g_2$ by applying the biholomorphism

$$\begin{aligned} \mathbb{A}^2 &\xrightarrow{\quad\sim\quad} \mathbb{A}^2 \\ (x,y) &\longmapsto \left(x - \tfrac{g_1(y)+g_2(y)}{2}, y\right). \end{aligned}$$

In particular, in this case, we have $d = o(g) = o(g_1) = o(g_2)$. Consider the ideals

$$\begin{aligned} I_i &= (x) + \mathfrak{m}^i && \text{for } i = 1,\dots,d-1, \\ J_j &= \left(x - \frac{j}{|j|}g(y)\right) + \mathfrak{m}^{|j|+d} && \text{for } -h_1+d \le j \le h_2-d \text{ and } j \neq 0, \\ J_0 &= (x) + \mathfrak{m}^d. \end{aligned}$$

Then, we can write the ideal $K = K_1 \cdot K_2$ as $K = I \cdot J$, where

$$I = \left(\prod_{i=1}^{d-1} I_i\right)^2, \qquad J = \left(\prod_{j=0}^{h_2-d} J_j\right)\cdot\left(\prod_{j=0}^{h_1-d} J_{-j}\right).$$

Let $\varepsilon_I : B_I = \mathrm{Bl}_I\,\mathbb{A}^2 \to \mathbb{A}^2$ be the blowup map. Then,

$$\mathrm{Bl}_K\,\mathbb{A}^2 = \mathrm{Bl}_{IJ}\,\mathbb{A}^2 \cong \mathrm{Bl}_{\varepsilon_I^{-1}(J)\cdot\mathscr{O}_{B_I}}\, B_I$$

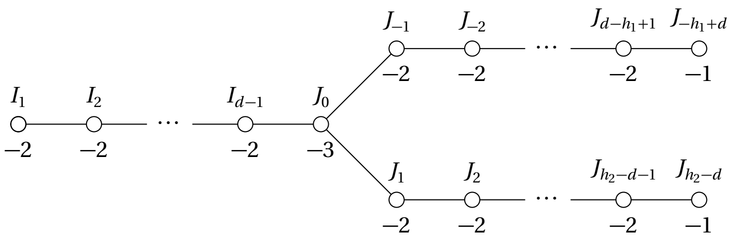
where the isomorphism is over $\mathbb{A}^2$. Notice that B_I is a toric variety. A direct computation in toric geometry shows that:

$$\varepsilon_I^{-1}(J)\cdot \mathscr{O}_{B_I} = \widetilde{\mathscr{I}}\cdot\widetilde{\mathscr{K}_1}\cdot\widetilde{\mathscr{K}_2}$$

where $\widetilde{\mathscr{I}}$ defines a Cartier divisor, while $\widetilde{\mathscr{K}_1}$ and $\widetilde{\mathscr{K}_2}$ are ideal sheaves of two 0-dimensional schemes with the same support $\{p\}\subset B_I$ with the property that p is a toric point. Consider a toric chart $U\subset B_I$ such that $U\cong\mathbb{A}^2$ and $p\in U$ is the origin. If a,b are toric coordinates on U, the two $\mathbb{C}[a,b]$-modules $\widetilde{\mathscr{K}_i}(U)$, for $i=1,2$ are complete towers of the form

$$\widetilde{K}_1 = \prod_{j=1}^{h_1-d+1}\big(a+\widetilde{g}(b)\big)+\mathfrak{m}_p^j, \qquad \widetilde{K}_2 = \prod_{j=1}^{h_2-d+1}\big(a-\widetilde{g}(b)\big)+\mathfrak{m}_p^j,$$

where $o(\widetilde{g}) = 1$ and $\mathfrak{m}_p = (a,b)$ is the ideal of the origin of U. Now, the property $o(\widetilde{g}) = 1$ implies that there is a biholomorphism around p which transforms the towers $\widetilde{K}_1$ and $\widetilde{K}_2$ in the monomial towers K_x and K_y in the first part of the statement. As a consequence, the Dynkin diagram of $\operatorname{Bl}_K\mathbb{A}^2$ is the following.



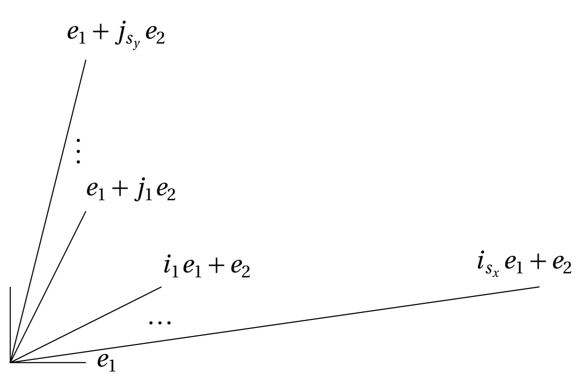
At this point, finding the Behrend number is a simple calculation analogous to those made in the proof of (1) and we leave it to the reader. □

Remark 2.3.18. Similarly as we have done in Remark 2.3.9, the above proposition can be easily generalised to non-complete towers. For example, in the easiest case when

$$K_x = \prod_{k=1}^{s_x}(x)+\mathfrak{m}^{i_k}, \qquad K_y = \prod_{k=1}^{s_y}(y)+\mathfrak{m}^{j_k},$$

for $1\le i_1<\cdots<i_{s_x}$ and $1\le j_1<\cdots<j_{s_y}$, are two monomial non-complete towers. Then, a fan Σ in $\mathbb{R}^2$ of the toric variety $X=\operatorname{Bl}_{K_xK_y}\mathbb{A}^2$ is the following.

$\Sigma =$



Moreover, the Behrend number $\nu_{\mathbb{C}[x,y]/K_x\cdot K_y}$ can be computed similarly as described in Remark 2.3.9.

Notice that, if the ray $\rho = e_1 + e_2$ belongs to the fan Σ then, X has only singularity of type A_n. While, if $\rho = e_1 + e_2 \notin \Sigma$ then, X has an isolated singularity of different kind associated to the cone $< e_1 + j_1 e_2, i_1 e_1 + e_2 >$. In particular, such singularity is never Gorenstein (see [13, Prop. 10.1.6.]), while the A_n singularities are always Gorenstein.

2.3.5 Behrend function and Hilbert–Samuel strata

We know by work of Briançon [8] and Iarrobino [36], that the punctual Hilbert scheme $\mathrm{Hilb}^n(\mathbb{A}^2)_0 \subset \mathrm{Hilb}^n(\mathbb{A}^2)$, parametrising subschemes entirely supported at the origin, contains the locus of the curvilinear schemes $\mathscr{C}_n$ as a Zariski open (and hence dense) subset. Moreover, the complement $\mathrm{Hilb}^n(\mathbb{A}^2)_0 \setminus \mathscr{C}_n$ can be stratified according to the *Hilbert–Samuel function*, also called the *type*, of fat points (see [36] for a definition). Let us consider the set-theoretic map

$$\beta_n \colon \mathrm{Hilb}^n(\mathbb{A}^2)_0 \to \mathbb{Z}, \qquad [I] \mapsto \nu_{\mathbb{C}[x,y]/I}.$$

We know by Example 2.2.5 this function is constantly equal to n on $\mathscr{C}_n \subset \mathrm{Hilb}^n(\mathbb{A}^2)_0$. Continuity of β_n is of course out of question. In fact, the following example shows that β_n is in general *not* even constant on the Hilbert–Samuel strata.

Example 2.3.19. Consider the two ideals

$$I = (xy, x^3 - y^3), \qquad J = (xy, x^4, y^3).$$

Then, $\mathrm{length}(\mathbb{C}[x,y]/I) = 6 = \mathrm{length}(\mathbb{C}[x,y]/J)$, and since I is a complete intersection we have also

$$\nu_{\mathbb{C}[x,y]/I} = 6$$

by Example 2.2.4. However, this is not the case for the Behrend number of the ideal J. Indeed, $J = (x, y^2)\cdot(x^3, y)$ is a product of curvilinear ideals and hence, in particular, a product of two towers. In order to compute the Behrend number of the ideal J we can proceed as suggested in Remark 2.3.18. Alternatively, we will show in the next section (see Section 2.4.2) an algorithm to compute the Behrend number of such kind of ideals. By applying it, one finds

$$\nu_{\mathbb{C}[x,y]/J} = 7 > 6.$$

Finally, we observe that they have the same type

$$T(I) = T(J) = (1,2,2,1,0,0,0),$$

and hence, they belong to the same Hilbert–Samuel stratum of $\mathrm{Hilb}^6(\mathbb{A}^2)_0$.

2.4 An algorithm for the Behrend number of a product of towers

In the previous section we computed the Behrend number of an arbitrary tower $K \subset \mathbb{C}[x,y]$ and of the product of two towers. In this section we explain an algorithmic procedure to perform the calculation for an arbitrary (finite) product of towers. This produces a huge number of examples of Behrend numbers of non-monomial schemes, analogously to Theorem 2.3.17 (2).

2.4.1 Products of towers: the complete case

We already observed (cf. Example 2.3.15) that the nodes of the Dynkin diagram attached to the blowup $\mathrm{Bl}_K \mathbb{A}^2$ along a complete tower $K \subset \mathbb{C}[x,y]$ can be labeled by ideals (see also Figure 2.3). In this section we shall construct more general Dynkin diagrams, each associated with a product of towers; in this subsection we focus on the complete case. In this context, all the Dynkin diagrams under consideration will have a node $c_{\mathfrak{m}}$ associated to the maximal ideal $\mathfrak{m} = (x,y) \subset \mathbb{C}[x,y]$, and all the other nodes will be connected to $c_{\mathfrak{m}}$ by a sequence of edges. In this section we will use the following terminology: we will say that $c_{\mathfrak{m}}$ has *level* 1, while the level of each other node c is defined as

$$\mathrm{level}(c) = 1 + \big|\text{number of edges separating } c \text{ from } c_{\mathfrak{m}}\big|.$$

For example, one can show that, if I is the ideal

$$I = \mathfrak{m} \cdot ((x) + \mathfrak{m}^2) \cdot ((y) + \mathfrak{m}^2) \cdot ((x+y) + \mathfrak{m}^2) \cdot ((x+y) + \mathfrak{m}^3), \tag{2.4.1}$$

then, the blowup $\mathrm{Bl}_I \mathbb{A}^2$ is smooth and the Dynkin diagram of $\mathrm{Exc}(\mathrm{Bl}_I \mathbb{A}^2)$ is as in Figure 2.5.

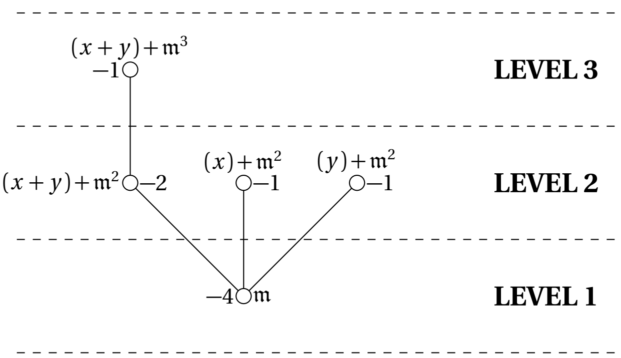


Figure 2.5. The Dynkin diagram of the ideal (2.4.1).

Note that, by construction of the Dynkin diagram associated to the exceptional locus of the blowup with centre the product of complete towers, a necessary condition for two nodes to be connected by an edge is that their levels differ by one unit. We will say that a node c_1 is a *descendant* of another node c_2 (or that c_2 is an *ancestor* of c_1) if c_1 and c_2 are connected by a sequence of edges and the level of c_1 is greater than the level of c_2.

Here is the general setup. Consider a set of complete towers

$$\mathcal{T} = \{K_1, \ldots, K_t\}$$

of the form

$$K_i = \prod_{k=1}^{s_i} (f_i) + \mathfrak{m}^k,$$

for $i = 1, \dots, t$, and consider also their product

$$T = \prod_{i=1}^{t} K_i.$$

Then, the blowup $\mathrm{Bl}_T \mathbb{A}^2$ is smooth and there is an algorithm to construct the Dynkin diagram of the exceptional locus of the map

$$\varepsilon\colon \mathrm{Bl}_T \mathbb{A}^2 \to \mathbb{A}^2.$$

Moreover, we will show how to compute, starting from such diagram, the Behrend number of the fat point $\mathbb{C}[x,y]/T$. We briefly describe the algorithm.

Let $h \geq 1$ be the maximum of the heights of the towers in $\mathcal{T}$, i.e.

$$h = \max\{\, s_i \mid i = 1, \dots, t \,\}.$$

Consider the h equivalence relations on $\mathcal{T}$ defined, for $r = 1, \dots, h$, by

$$K_i \sim_r K_j \quad \Leftrightarrow \quad \begin{cases} 1 \leq r \leq \min\{s_i, s_j\}, \ f_i \equiv f_j \mod \mathfrak{m}^r, \text{or} \\ r > \max\{s_i, s_j\}, \end{cases}$$

and call classes in excess the classes of the form $[K_i]^{\sim_r}$ for $r > s_i$. In particular, there is at most one class in excess for any $r = 1, \dots, h$.

We are now ready to construct the underlying graph of the Dynkin diagram for $\mathrm{Exc}(\mathrm{Bl}_T \mathbb{A}^2)$. We put one node at the first level, namely the node $c_{\mathfrak{m}}$ corresponding to the unique class in $\mathscr{T}/\sim_1$, and, at the i-th level, we put a node for each element in $\mathcal{T}/\sim_i$ excluding the possible class in excess. Finally, we add an edge joining the node associated to some class $[K]^{\sim_r}$ to the node associated to some class $[K']^{\sim_{r'}}$ if and only if

$$\left| r - r' \right| = 1 \text{ and } [K]^{\sim_r} \cap [K']^{\sim_{r'}} \neq \emptyset.$$

The self-intersection at each node of level strictly greater then one is given by

$$-\left|\{\text{edges issuing from the node}\}\right|$$

while, the node $c_{\mathfrak{m}}$ is labeled by the self intersection

$$-\left|\{\text{edges issuing from } c_{\mathfrak{m}}\}\right| - 1.$$

This allows to compute the multiplicities of the irreducible components of the exceptional divisor of $\mathrm{Bl}_T \mathbb{A}^2$. We now briefly explain how to compute Behrend numbers.

Let I be an ideal appearing as a factor of some tower in $\mathscr{T}$, an let c_I be the node of the Dynkin diagram associated to I (see Example 2.3.15). Let also c be any node in the Dynkin diagram and let D_c be the corresponding irreducible component of $\mathrm{Exc}(\mathrm{Bl}_T \mathbb{A}^2)$. Then, the contribution of the ideal I to the multiplicity of the exceptional divisor along the component D_c is given by

$$\begin{cases} \text{level of } c_I & \text{if } c = c_I \text{ or if } c \text{ is a descendant of } c_I, \\ \text{level of } c & \text{if } c \text{ is an ancestor of } c_I, \\ 1 & \text{otherwise.} \end{cases}$$

Now, summing up all these contributions over all pairs (I, c), one obtains the Behrend number of $\mathbb{C}[x,y]/T$.

2.4.2 Products of towers: the non-complete case

Once more, the complete case helps us in understanding the non-complete case.

Let

$$\mathcal{T} = \{ K_1, \dots, K_t \}$$

be a set of towers of the form

$$K_i = \prod_{k=1}^{s_i} (f_i) + \mathfrak{m}^{i_k},$$

for $i = 1, \dots, t$, and $1 \le i_1 < \cdots < i_{s_i}$, and let

$$T = \prod_{i=1}^{t} K_i$$

be their product. Consider also the set of complete towers

$$\widetilde{\mathcal{T}} = \{ \widetilde{K}_1, \dots, \widetilde{K}_t \}$$

defined by

$$\widetilde{K}_i = \prod_{k=1}^{i_{s_i}} (f_i) + \mathfrak{m}^k,$$

for $i = 1, \dots, t$, and let us set

$$\widetilde{T} = \prod_{i=1}^{t} \widetilde{K}_i.$$

Let us also call $\varepsilon\colon B = \mathrm{Bl}_T \mathbb{A}^2 \to \mathbb{A}^2$ and $\widetilde{\varepsilon}\colon \widetilde{B} = \mathrm{Bl}_{\widetilde{T}} \mathbb{A}^2 \to \mathbb{A}^2$ the blowup maps. Then, $\widetilde{\varepsilon}^{-1}(T) \cdot \mathscr{O}_{\widetilde{B}}$ defines a Cartier divisor. Hence, there is a canonical $\mathbb{A}^2$-morphism $\varphi\colon \widetilde{B} \to B$. Let

$$\mathrm{Exc}(\widetilde{B}) = \widetilde{C}_1 \cup \cdots \cup \widetilde{C}_\alpha \subset \widetilde{B}, \qquad \mathrm{Exc}(B) = C_1 \cup \cdots \cup C_\beta \subset B$$

be the decompositions of $\mathrm{Exc}(\widetilde{B})$ and $\mathrm{Exc}(B)$ into irreducible components. Clearly, we have $\alpha \ge \beta$. Then, as per Theorem 0.9.2, the morphism φ must contract some of the curves in $\mathrm{Exc}(\widetilde{B})$ and it is an isomorphism when restricted to the complement of the contracted curves. In particular, up to reordering the components of $\mathrm{Exc}(\widetilde{B})$, the map

$$\varphi\big|_{\widetilde{B} \smallsetminus (\widetilde{C}_{\beta+1} \cup \widetilde{C}_{\beta+2} \cup \cdots \cup \widetilde{C}_\alpha)} \colon \widetilde{B} \smallsetminus (\widetilde{C}_{\beta+1} \cup \widetilde{C}_{\beta+2} \cup \cdots \cup \widetilde{C}_\alpha) \to B \smallsetminus \varphi(\widetilde{C}_{\beta+1} \cup \widetilde{C}_{\beta+2} \cup \cdots \cup \widetilde{C}_\alpha)$$

is an isomorphism which restricts, for $j = 1, \dots, \beta$, to an isomorphism

$$\widetilde{C}_j \smallsetminus (\widetilde{C}_{\beta+1} \cup \widetilde{C}_{\beta+2} \cup \cdots \cup \widetilde{C}_\alpha) \to C_j \smallsetminus \varphi(\widetilde{C}_{\beta+1} \cup \widetilde{C}_{\beta+2} \cup \cdots \cup \widetilde{C}_\alpha).$$

Therefore, for $j = 1,\dots,\beta$, the multiplicity of the exceptional divisor $E_T\mathbb{A}^2 = V(\varepsilon^{-1}(T)\cdot\mathscr{O}_B)$ along C_j equals the multiplicity of the Cartier divisor defined by $\widetilde{\varepsilon}^{-1}(T)\cdot\mathscr{O}_{\widetilde{B}}$ along $\widetilde{C}_j$.

This observation allows one to compute the Behrend number of any (finite) product of towers.

2.4.3 Examples

In general, one does not need to pass trough the blowup of a complete tower to compute the Behrend number of some tower K. The convenience in introducing the complete towers even in the non-complete case is in the computations. The following example should explain the situation.

Example 2.4.1. Consider the ideal $I = (x, y^2) \subset \mathbb{C}[x, y]$ and the blowup $\varepsilon\colon B_I = \mathrm{Bl}_I\,\mathbb{A}^2 \to \mathbb{A}^2$. Then, B_I is described in Proposition 2.3.3 and it is a toric surface covered by two charts: the first, U_1, is isomorphic to the affine quadric cone, whereas the second, U_2, is smooth and isomorphic to $\mathbb{A}^2$. In particular, U_1 is the affine toric variety described by the cone generated by the rays of primitive vectors $\rho_1 = e_2, \rho_2 = 2e_1 + e_2$ and, by standard toric geometry (see [13, § 3.1]), we have an isomorphism of $\mathbb{A}^2$-schemes

$$U_1 \cong \operatorname{Spec}\mathbb{C}[x, x^{-1}y^2, y].$$

If we introduce the variables $a = x, b = x^{-1}y^2, c = y$, then the restriction to U_1 of the blowup map is associated to the $\mathbb{C}$-algebra homomorphism

$$\begin{array}{ccc} \mathbb{C}[x,y] & \longrightarrow & S = \mathbb{C}[a,b,c]/(ab-c^2) \\ x & \longmapsto & a \\ y & \longmapsto & c. \end{array}$$

Computing $\varepsilon^{-1}(I)\cdot\mathscr{O}_{U_1}$, one finds

$$\varepsilon^{-1}(I)\cdot\mathscr{O}_{U_1} = (a, c^2) = (a, ab) = (a) \subset S,$$

so that one would be tempted to conclude that $\nu_{\mathbb{C}[x,y]/I} = 1$. This is, in fact, incorrect, because, in the local ring $\mathscr{O}_{U_1,\mathrm{Exc}(\varepsilon)} \cong S_{(a,c)}$ the function b is invertible and we have $\nu_{\mathbb{C}[x,y]/I} = \mathrm{ord}_{\mathrm{Exc}(\varepsilon)}(a) = \mathrm{ord}_{\mathrm{Exc}(\varepsilon)}(c^2b^{-1}) = \mathrm{ord}_{\mathrm{Exc}(\varepsilon)}(c^2) = 2\cdot\mathrm{ord}_{\mathrm{Exc}(\varepsilon)}(c) = 2$.

This complication never occurs in the case of smooth surfaces.

Even though the procedure described in Section 2.4.2 above is quite straightforward, one may need to use a computer to actually compute the Behrend number of an arbitrary (finite) product of towers. Computing the length of a product of towers, on the other hand, can often be quite complicated. Below we show an example that can be computed by hand.

Example 2.4.2. Let $K = \prod_{1\le k\le s}(x) + \mathfrak{m}^{i_k} = \prod_{1\le k\le s}(x, y^{i_k})$ be a monomial tower with $1 < i_1 < \cdots < i_s$, and set $J = K\cdot\mathfrak{m}^n$ for some integer $n > 0$. Then, the following formula holds

$$J = (x,y)^{n+s} \cap \left(x^s,\, x^{s-i}\, y^{n+\sum_{k=1}^{i} i_k} \;\middle|\; i = 1,\dots,s \right).$$

As the ideal has now taken on a more pleasant form, the following formulas can easily be obtained:

$$\ell_{\mathbb{C}[x,y]/J} = \ell_{\mathbb{C}[x,y]/K} + \frac{n(n+1)+2ns}{2},$$
$$\nu_{\mathbb{C}[x,y]/J} = \nu_{\mathbb{C}[x,y]/K} + sn + n + s.$$

2.5 The general normal case

In this section, we will completely solve the problem of computing the Behrend number $\nu_{\mathbb{C}[x,y]/I}$ for a normal monomial ideal $I \subset \mathbb{C}[x,y]$. Moreover, in Theorem 2.6.4 we will give a toric description of the blowup $\mathrm{Bl}_I \mathbb{A}^2$ and we will prove a factorisation theorem (Corollary 2.6.11) that allows one to write the ideal I uniquely as a product of powers of much easier ideals, namely the normalisations of the monomial complete intersection ideals (x^h, y^k).

2.5.1 The key example

Consider the ideals

$$I = x^2 + \mathfrak{m}^3 = (x^2, xy^2, y^3), \qquad J = \mathfrak{m}\cdot((x)+\mathfrak{m}^2)\cdot I. \tag{2.5.1}$$

In particular $\ell_{\mathbb{C}[x,y]/I} = 5$ and $\ell_{\mathbb{C}[x,y]/J} = 14$.

We want to perform, for the ideals I and J, the same analysis that we did for the ideals in the previous sections. We have (for instance via Proposition 0.2.2)

$$X_I = \mathrm{Bl}_I \mathbb{A}^2 = \left\{ ((x,y),[w_0:w_1:w_2]) \in \mathbb{A}^2 \times \mathbb{P}^2 \;\middle|\; \begin{array}{c} x w_1 = y w_2 \\ y^2 w_0 = x w_2 \\ y w_0 w_1 = w_2^2 \end{array} \right\}$$

and the exceptional locus is

$$D_I = \mathrm{Exc}(X_I) = \left\{ ((x,y),[w_0:w_1:w_2]) \in X_I \;\middle|\; x = y = w_2 = 0 \right\} \cong \mathbb{P}^1.$$

In order to study the variety X_I we cover it with the three affine charts

$$X_{I,i} = X_I \cap (\mathbb{A}^2 \times \{w_i \neq 0\}), \quad i = 0,1,2.$$

We will also denote by $D_{I,i}$, for $i = 0,1$, the chart on D_I given by $D_{I,i} = D_I \cap X_{I,i}$.

Now, $X_{I,2}$ is smooth, while $X_{I,0}$ and $X_{I,1}$ have each an isolated singular point $p_i \in D_{I,i}$ for $i = 0,1$. The singular charts have the form

$$X_{I,0} \cong \mathbb{A}^2/G_0, \quad X_{I,1} \cong \mathbb{A}^2/G_1,$$

where, given a primitive third root of unity $\xi_3 \in \mathbb{C}^\times$, the groups G_0, G_1 are

$$G_0 = \left\langle \begin{pmatrix} \xi_3 & 0 \\ 0 & \xi_3 \end{pmatrix} \right\rangle \subset \mathrm{GL}(2,\mathbb{C}), \qquad G_1 = \left\langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle \subset \mathrm{SL}(2,\mathbb{C}).$$

As a consequence, the surface X_I is normal. This is a general fact about quotient surface singularities and it can also be deduced from Proposition 0.10.2.

Let $\varepsilon_I : X_I \to \mathbb{A}^2$ be the blowup map and let

$$\varphi : \widetilde{X}_I \to X_I$$

be the minimal resolution of X_I, i.e. $\widetilde{X}_I$ is a smooth surface and φ is a projective birational morphism which does not contract any rational (-1)-curve. It is well known that the variety $\widetilde{X}_I$

is obtained by blowing up the singular points p_0 and p_1. Let us also denote by $\widetilde{D}_I \subset \mathrm{Exc}(\varepsilon_I \circ \varphi) \subset \widetilde{X}_I$ the strict transform of D_I, i.e. the Zariski closure of $\varphi^{-1}(D_I \setminus \{p_0, p_1\})$. Given the description of the singularities, we have that $\mathrm{Exc}(\varphi)$ is a disjoint union of two smooth projective rational curves L_0, L_1, corresponding respectively to p_0 and p_1, each of which intersects the line $\widetilde{D}_I$ at a point. Furthermore, the self-intersections of L_0 and L_1 are respectively

$$L_0^2 = -3, \quad L_1^2 = -2.$$

Notice that the map $\varepsilon_I \circ \varphi$ is a projective birational morphism of smooth surfaces and hence, by classical theory of surfaces (see [3, Ch. III]), it follows that $\widetilde{X}_I$ contains a smooth rational projective (-1)-curve, and the only possible such curve is $\widetilde{D}_I$. The Dynkin diagram attached to $\{\widetilde{D}_I, L_0, L_1\}$ is depicted in Figure 2.6.

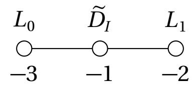


Figure 2.6. The Dynkin diagram attached to $\{\widetilde{D}_I, L_0, L_1\}$.

We claim that there is a canonical isomorphism of $\mathbb{A}^2$-schemes between $X_J = \mathrm{Bl}_J \mathbb{A}^2$ and $\widetilde{X}_I$, where J is as in (2.5.1). Thanks to Lemma 0.2.1, we know that there is a canonical morphism of $\mathbb{A}^2$-schemes

$$\psi \colon X_J \to X_I.$$

This follows from the existence of the isomorphism of $\mathbb{A}^2$-schemes

$$X_J \xrightarrow{\sim} \mathrm{Bl}_{\varepsilon^{-1}(I)\cdot \mathscr{O}_B} B$$

where $\varepsilon \colon B \to \mathbb{A}^2$ is the blowup with centre the ideal $\mathfrak{m} \cdot (x, y^2)$, together with the universal property of blowups. In fact, we observe that there are canonical isomorphisms of $\mathbb{A}^2$-schemes

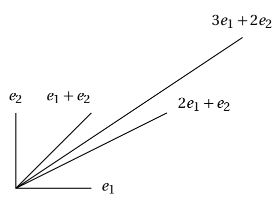
$$\begin{tikzcd} & Z \arrow[dl, "\widetilde{\vartheta}"'] \arrow[dr, "\vartheta_J"] & \\ \widetilde{X}_I & & X_J \end{tikzcd}$$

where Z is the result of an iterated blowup, namely

$$Z \xrightarrow{\mu} B \xrightarrow{\varepsilon} \mathbb{A}^2$$

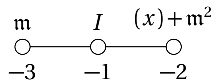
where μ is the blowup of B with centre the intersection point of the two irreducible components of $\mathrm{Exc}(\varepsilon)$.

In order to construct the isomorphisms $\widetilde{\vartheta}$ and ϑ_J, we start by noticing that the surface Z just described is the toric variety associated to the fan Σ in $\mathbb{R}^2$ shown below.



Now, the same computations as those we did in the previous section show that $(\varepsilon\circ\mu)^{-1}(I)\cdot\mathscr{O}_Z$ and $(\varepsilon\circ\mu)^{-1}(J)\cdot\mathscr{O}_Z$ define Cartier divisors on Z and, as a consequence, there exist canonical $\mathbb{A}^2$-morphisms $\vartheta_I\colon Z\to X_I$ and $\vartheta_J\colon Z\to X_J$. Moreover, ϑ_I does not contract any (-1)-curve and, as a consequence, it lifts to an isomorphism $\widetilde{\vartheta}\colon Z\to\widetilde{X}_I$ because of the universal property of the minimal resolution (see [3, Thm. (6.2)]). The map ϑ_J is an isomorphism because $\varepsilon^{-1}(I)\cdot\mathscr{O}_B$ is the product of a principal (Cartier) ideal sheaf times the ideal sheaf of the reduced intersection point of the two irreducible components of $\mathrm{Exc}(\varepsilon)$.

As a consequence, X_J and $\widetilde{X}_I$ are canonically isomorphic and, if we label their Dynkin diagram (see Figure 2.6) as explained in Example 2.3.15, then we obtain the following diagram.



Notice that, the Dynkin diagram above is different from those we encountered in Theorem 2.3.17 or appearing in Section 2.4.

Now we move to the computation of the Behrend numbers $\nu_{\mathbb{C}[x,y]/I}$ and $\nu_{\mathbb{C}[x,y]/J}$ exploiting the canonical isomorphisms of $\mathbb{A}^2$-schemes just described. The computation of $\nu_{\mathbb{C}[x,y]/J}$ is achieved, just as in the proof of Theorem 2.3.11, via toric geometry, yielding the answer

$$\nu_{\mathbb{C}[x,y]/J}=21.$$

In order to compute $\nu_{\mathbb{C}[x,y]/I}$, we start by noticing that the morphism $\varphi\circ\widetilde{\vartheta}\colon Z\to X_I$ contracts two disjoint smooth rational projective curves over two distinct points of X_I and it is an isomorphism outside such curves. Therefore, if $C\subset Z$ is the curve that dominates $\mathrm{Exc}(X_I)$, then $\varphi\circ\widetilde{\vartheta}|_C$ is a birational morphism and we have

$$\nu_{\mathbb{C}[x,y]/I}=\mathrm{mult}_C(V((\varepsilon\circ\mu)^{-1}(I)\cdot\mathscr{O}_Z)).$$

Again, toric geometry applied as in the proof of Theorem 2.3.11 gives the answer, namely

$$\nu_{\mathbb{C}[x,y]/I}=6.$$

In Theorem 2.6.5, we shall describe a general procedure which, in particular, allows one to compute the number $\nu_{\mathbb{C}[x,y]/I}$.

2.5.2 Behrend number and factorisations of normal ideals

Notation 2.6. Set $I_{h,k}=(x^h,y^k)\subset\mathbb{C}[x,y]$. Then we let $\mathfrak{n}_{h,k}=\overline{I}_{h,k}$ be the normalisation of $I_{h,k}$, defined as in Proposition 0.10.2.

Example 2.6.1. For istance, $\mathfrak{n}_{h,h}=\mathfrak{m}^h$ for all $h\geq 0$. One also has $\mathfrak{n}_{2,3}=(x^2,xy^2,y^3)$.

Lemma 2.6.2. *For any $\delta\geq 0$ and $h,k>0$, there is an identity of ideals*

$$\mathfrak{n}_{h,k}^{\delta}=\mathfrak{n}_{\delta h,\delta k}\subset\mathbb{C}[x,y].$$

Proof. This is trivial for $\delta = 0, 1$, and it follows, for higher δ, combining Proposition 0.10.2 with the general formula

$$\mathrm{Conv}_{\mathbb{Q}}(\delta v_1, \dots, \delta v_s) = \mathrm{Conv}_{\mathbb{Q}}\left(\sum_{i=1}^{s} n_i v_i \;\middle|\; \sum_{i=1}^{s} n_i = \delta,\ n_i \geq 0 \right) \subset V$$

for any choice of vectors $v_1, \dots, v_s \in V$ in some $\mathbb{Q}$-vector space V. □

Lemma 2.6.3. *Let X be the toric surface with fan Σ in $\mathbb{R}^2$ generated by the primitive vectors*

$$\rho_0 = e_1, \quad \rho_1 = \beta e_1 + \alpha e_2, \quad \rho_2 = e_2.$$

Then X and $\mathrm{Bl}_{\mathfrak{n}_{\alpha,\beta}} \mathbb{A}^2$ are canonically isomorphic as $\mathbb{A}^2$-schemes.

Proof. The variety X is, by construction covered by two charts U_1 and U_2 respectively associated to the cones $\sigma_1 = \langle \rho_0, \rho_1 \rangle$ and $\sigma_2 = \langle \rho_1, \rho_2 \rangle$. In particular, by standard toric geometry (see [13, § 3.1]), there exist two integers $s_1, s_2 \geq 0$ and Laurent monomials $m_{1,1}, \dots, m_{1,s_1}, m_{2,1}, \dots, m_{2,s_2} \in \mathbb{C}(x, y)$ (the cone σ_i is smooth if and only if $s_i = 0$ and no Laurent monomial is needed) such that

$$\begin{aligned} U_1 &= \operatorname{Spec} \mathbb{C}\left[x^\alpha y^{-\beta}, y, m_{1,1}, \dots, m_{1,s_1}\right] \\ U_2 &= \operatorname{Spec} \mathbb{C}\left[x, x^{-\alpha} y^\beta, m_{2,1}, \dots, m_{2,s_2}\right]. \end{aligned}$$

Let us denote by $S_1 = \mathbb{C}[x^\alpha y^{-\beta}, y, m_{1,1}, \dots, m_{1,s_1}]$ and $S_2 = \mathbb{C}[x, x^{-\alpha} y^\beta, m_{2,1}, \dots, m_{2,s_2}]$ the affine rings of U_1 and U_2 and by $\varepsilon \colon X \to \mathbb{A}^2$ the structure morphism. Then, if $I = (x^\alpha, y^\beta)$, we have

$$\varepsilon\big|_{U_1}^{-1}(I) \cdot \mathscr{O}_{U_1} = (y^\beta) \subset S_1, \qquad \varepsilon\big|_{U_2}^{-1}(I) \cdot \mathscr{O}_{U_2} = (x^\alpha) \subset S_2,$$

which implies that the sheaf $\varepsilon^{-1}(I) \cdot \mathscr{O}_X$ defines a Cartier divisor on X.

As a consequence we have a cononical birational morphism of $\mathbb{A}^2$-schemes $\psi \colon X \to \mathrm{Bl}_I \mathbb{A}^2$. If this morphism is finite then, by Proposition 0.9.1, it must coincide with the normalisation morphism and this would provide an isomorphism of $\mathrm{Bl}_I \mathbb{A}^2$-schemes between X and $\mathrm{Bl}_{\mathfrak{n}_{\alpha\beta}} \mathbb{A}^2$ (which in particular is an isomorphism of $\mathbb{A}^2$-schemes). Finally, the morphism ψ is finite because it is proper and has finite fibres. Indeed, it is an isomorphism away from the exceptional loci $\mathrm{Exc}(X)$ and $\mathrm{Exc}(\mathrm{Bl}_I \mathbb{A}^2)$, and it is a dominant morphism between irreducible projective curves when restricted to the exceptional loci. □

Theorem 2.6.4. *Let X be a toric surface which admits a fan Σ_X in $\mathbb{R}^2$ that covers the first quadrant $\mathbb{R}^2_{\geq 0}$ i.e. Σ_X is generated by the rays with primitive vectors $\rho_0 = e_1, \rho_{r+1} = e_2, \rho_k = m_k e_1 + n_k e_2$, for $k = 1, \dots, r$, where $m_k, n_k > 0$ and $\gcd(m_k, n_k) = 1$. Then, there is a canonical isomorphism*

$$X \xrightarrow{\ \sim\ } \mathrm{Bl}_I \mathbb{A}^2$$

where $I = \prod_{1 \leq k \leq r} \mathfrak{n}_{n_k, m_k}$.

Proof. The statement follows by applying Lemma 2.6.3 and Lemma 0.2.1. □

Theorem 2.6.5. *Let $\alpha, \beta > 0$ be two positive integers. Then,*

$$\nu_{\mathbb{C}[x,y]/\mathfrak{n}_{\alpha,\beta}} = \frac{\alpha \cdot \beta}{\gcd(\alpha, \beta)}.$$

Proof. Suppose first that $\gcd(\alpha,\beta)=1$. Then, Euclid's algorithm provides two positive integers $h,k\in\mathbb{N}$ such that

$$k\beta-h\alpha=1.$$

Let J be the ideal

$$J=\mathfrak{n}_{\alpha,\beta}\cdot\mathfrak{n}_{k,h}.$$

Theorem 2.6.4 implies that there exists an open affine subset $U\subset \mathrm{Bl}_J\mathbb{A}^2$, isomorphic to $\mathbb{A}^2$, such that

$$U\cong\begin{cases}\mathrm{Spec}(\mathbb{C}[x^{-\alpha}y^{\beta},x^ky^{-h}]) & \text{if } \frac{\alpha}{\beta}<\frac{k}{h}\\ \mathrm{Spec}(\mathbb{C}[x^{\alpha}y^{-\beta},x^{-k}y^{h}]) & \text{if } \frac{\alpha}{\beta}>\frac{k}{h}.\end{cases}$$

We put $\alpha/\beta>k/h$, the other case being identical. If we denote by $\varepsilon\colon \mathrm{Bl}_J\mathbb{A}^2\to\mathbb{A}^2$ the blowup map, and by $s=x^{\alpha}y^{-\beta}, t=x^{-k}y^{h}$ the affine coordinates on U, then the restriction of the blowup map to U is given by

$$\begin{aligned} U &\xrightarrow{\ \varepsilon|_U\ } \mathbb{A}^2\\ (s,t) &\longmapsto (s^ht^{\beta},s^kt^{\alpha}).\end{aligned}$$

By construction, the intersection $\mathrm{Exc}(U)=\mathrm{Exc}(\varepsilon)\cap U$ consists of the two coordinate axes of U. In particular, given the natural map $\lambda\colon \mathrm{Bl}_J\mathbb{A}^2\to \mathrm{Bl}_{\mathfrak{n}_{\alpha,\beta}}\mathbb{A}^2$, the strict transform of $\mathrm{Exc}(\mathrm{Bl}_{\mathfrak{n}_{\alpha,\beta}}\mathbb{A}^2)$ via $\lambda|_U$ is the irreducible component of $\mathrm{Exc}(U)$ given by $C=V(t)$. This implies that

$$\nu_{\mathbb{C}[x,y]/\mathfrak{n}_{\alpha,\beta}}=\mathrm{mult}_C\left(\varepsilon|_U^{-1}(\mathfrak{n}_{\alpha,\beta})\cdot\mathscr{O}_U\right)=\alpha\cdot\beta,$$

where, the first equality follows from the fact that λ is an isomorphism away from its exceptional locus $\mathrm{Exc}(\lambda)$ and the second follows from an easy computation.

Suppose now that $\gcd(\alpha,\beta)=\delta>1$. Then, by Lemma 2.6.2 and Proposition 2.2.8, we have

$$\nu_{\mathbb{C}[x,y]/\mathfrak{n}_{\alpha,\beta}}=\nu_{\mathbb{C}[x,y]/\mathfrak{n}^{\delta}_{\alpha',\beta'}}=\delta\cdot\nu_{\mathbb{C}[x,y]/\mathfrak{n}_{\alpha',\beta'}}=\delta\cdot\alpha'\cdot\beta',$$

where $\alpha'=\alpha/\delta$ and $\beta'=\beta/\delta$, i.e.

$$\nu_{\mathbb{C}[x,y]/\mathfrak{n}_{\alpha,\beta}}=\frac{\alpha\cdot\beta}{\delta},$$

as required. □

Remark 2.6.6. Exploiting toric geometry techniques and the isomorphism of Theorem 2.6.4, one can generalise the computation of $\nu_{\mathbb{C}[x,y]/\mathfrak{n}_{\alpha,\beta}}$ to an arbitrary normal monomial ideal, along the lines of the example fully worked out in Section 2.5.1.

Proposition 2.6.7. *Let I be the ideal generated by the monomials*

$$x^{a_0},x^{a_1}y^{b_{n-1}},\dots,x^{a_i}y^{b_{n-i}},\dots,x^{a_{n-1}}y^{b_1},y^{b_0},$$

where $a_i>a_{i+1}, b_i>b_{i+1}$ and we also put $a_n=b_n=0$. Suppose that I is normal.

Let $0=i_0<\dots<i_t=n$ be the strictly increasing sequence of positive integers such that

$$v_k=(a_{i_k},b_{n-i_k}),\ \textit{for}\ k=0,\dots,t,$$

are the vertices of ∂Q_I *(see Remark 0.10.6); then*

$$I = \prod_{k=1}^{t} \mathfrak{n}_{a_{i_{k-1}}-a_{i_k}, b_{n-i_k}-b_{n-i_{k-1}}}. \tag{2.6.1}$$

Proof. Let us set

$$J = \prod_{k=1}^{t} \mathfrak{n}_{a_{i_{k-1}}-a_{i_k}, b_{n-i_k}-b_{n-i_{k-1}}},$$

and let $Q_I, Q_J \subset \mathbb{Q}^2$ be defined as in Proposition 0.10.2. Then, the blowup $\mathrm{Bl}_J \mathbb{A}^2$ is a normal surface, as per Remark 0.10.3, and therefore the claim is equivalent to the equality

$$Q_I = Q_J.$$

Since, in general, we have $Q_{\mathfrak{n}_{\alpha,\beta}} = \mathrm{Conv}_{\mathbb{Q}}((\alpha,0),(0,\beta)) + \mathbb{Q}^2_{\geq 0}$, we also have

$$Q_J = \mathrm{Conv}_{\mathbb{Q}}(A) + \mathbb{Q}^2_{\geq 0}$$

where

$$A = \left\{ (a_0,0) + \sum_{j\in\Delta} [(a_{i_j}, b_{n-i_j}) - (a_{i_{j-1}}, b_{n-i_{j-1}})] \;\middle|\; \Delta \subset \{1,\ldots,t\} \right\}.$$

Notice that $Q_I \subset Q_J$ because $v_0 \in A$ and

$$v_k = (a_0,0) + \sum_{j=1}^{k} \left[(a_{i_j}, b_{n-i_j}) - (a_{i_{j-1}}, b_{n-i_{j-1}}) \right] \in A$$

for all $k = 1,\ldots,t$. On the other hand, the inclusion $A \subset Q_I$ is an easy consequence of the convexity of Q_I and it implies $Q_J \subset Q_I$. □

Example 2.6.8. Let $I = (x^6, x^4y, x^2y^2, xy^3, y^5)$ be the same ideal as in Example 0.10.7. Then, I is normal and it factors as

$$I = \mathfrak{n}_{1,2} \cdot \mathfrak{n}_{1,1} \cdot \mathfrak{n}_{2,1}^2.$$

Remark 2.6.9. Thanks to the celebrated Pick's theorem on lattice polygons, we can compute the Behrend number of the ideals of the form $\mathfrak{n}_{\alpha,\beta}$ as well as the length of normal ideals I given as in Equation (2.6.1). In particular, we have

$$\ell_{\mathbb{C}[x,y]/\mathfrak{n}_{\alpha,\beta}} = \frac{\alpha\beta + \alpha + \beta - \gcd(\alpha,\beta)}{2}$$

and, for I as in Equation (2.6.1),

$$\ell_{\mathbb{C}[x,y]/I} = \frac{a_0 + b_0 + \sum_{k=1}^{t} \left[\det\begin{pmatrix} a_{i_{k-1}} & b_{n-i_{k-1}} \\ a_{i_k} & b_{n-i_k} \end{pmatrix} - \gcd(a_{i_{k-1}} - a_{i_k}, b_{n-i_k} - b_{n-i_{k-1}}) \right]}{2}.$$

Corollary 2.6.10. *Let* $I \subset \mathbb{C}[x,y]$ *be the ideal* (2.6.1) *appearing in Proposition 2.6.7. Then* $\mathrm{Bl}_I \mathbb{A}^2$ *is canonically isomorphic, as an* $\mathbb{A}^2$*-scheme, to the toric surface whose fan is generated by the primitive vectors*

$$\rho_0, \ldots, \rho_{t+1}$$

defined by $\rho_0 = e_1$, $\rho_{t+1} = e_2$ *and*

$$\rho_k = \beta_k \cdot e_1 + \alpha_k \cdot e_2 \text{ for } k = 1, \dots, t$$

where, if $\delta_k = \gcd(a_{i_{k-1}} - a_{i_k}, b_{n-i_k} - b_{n-i_{k-1}})$, *then*

$$\alpha_k = \frac{a_{i_{k-1}} - a_{i_k}}{\delta_k} \text{ and } \beta_k = \frac{b_{n-i_k} - b_{n-i_{k-1}}}{\delta_k}.$$

In particular, there is a bijective correspondence

$$\left\{ \Sigma \text{ fan in } N \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^2 \;\middle|\; \begin{array}{c} N \cong \mathbb{Z}^2, \\ \mathrm{Supp}(\Sigma) = \mathbb{R}^2_{\geq 0} \end{array} \right\} \xleftrightarrow{1:1} \left\{ I = \prod_{k=1}^{t} \mathfrak{n}_{\alpha_k, \beta_k} \;\middle|\; \begin{array}{c} (\alpha_i, \beta_i) \neq (\alpha_j, \beta_j) \text{ for } i \neq j, \\ \gcd(\alpha_i, \beta_i) = 1 \end{array} \right\}.$$

We note that Proposition 2.6.7 can be interpreted as a factorisation statement, as follows.

Corollary 2.6.11. *Let* $\mathfrak{N}$ *be the set of normal monomial ideals in* $\mathbb{C}[x, y]$. *Then, every* $I \in \mathfrak{N}$ *factors as a product of ideals in* $\mathfrak{N}$,

$$I = \prod_{k=1}^{t} \mathfrak{n}_{\alpha_k, \beta_k}^{\delta_k}, \tag{2.6.2}$$

where $\delta_k \geq 1$ *and* $\gcd(\alpha_k, \beta_k) = 1$ *for* $k = 1, \dots, t$. *Such factorisation is unique up to reordering the factors.*

A similar property cannot be expected to hold on a larger class of ideals than $\mathfrak{N}$. For instance, as mentioned in Remark 0.10.3, if we drop the normality assumption we have, for the same ideal, two factorisations $\mathfrak{m}^3 = \mathfrak{m} \cdot (x^2, y^2)$, where (x^2, y^2) is not normal.

Combining Lemma 2.6.2, Proposition 2.2.8, and Theorem 2.6.4 with one another, we also obtain the following correspondence.

Theorem 2.6.12. *Let* $I \subset \mathbb{C}[x, y]$ *be a normal monomial ideal of finite colength. There is a bijective correspondence*

$$\left\{ \begin{array}{c} \text{ideals } \mathfrak{n}^{\delta}_{\alpha,\beta} \text{ appearing in the} \\ \text{factorisation (2.6.2) of } I \end{array} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{c} \text{irreducible} \\ \text{components of } E_I \mathbb{A}^2 \end{array} \right\}.$$

In particular, if $J \subset \mathbb{C}[x, y]$ *is an arbitrary monomial ideal and* $I = \overline{J}$ *is its normalisation, then* $E_J \mathbb{A}^2$ *has at most* t *irreducible components, where* t *is as in Equation* (2.6.2).

Remark 2.6.13. Corollary 2.6.11 and Theorem 2.6.12 can be seen as explicit instances of the factorisation theory developed by Lipman [48, Section V].

2.7 The Behrend function of a fat point via normalisation

In this section we will prove a formula (see Theorem 2.7.2 below) for the Behrend number of a nonnormal monomial ideal in terms of the Behrend number of its normalisation and the normalisation map.

Let $I \subset \mathbb{C}[x,y]$ be a monomial ideal of finite colength. When $\mathrm{Bl}_I \mathbb{A}^2$ is not normal, the computation of the Behrend number of I poses some difficulties, but the main result in this section resolves them explicitly. More precisely, we shall prove a general formula for the Behrend number

$$\nu_{\mathbb{C}[x_1,\ldots,x_N]/I} = \sum_{D \subset E_I \mathbb{A}^N} \mathrm{mult}_D\left(E_I \mathbb{A}^2\right) \tag{2.7.1}$$

of an arbitrary fat point $I \subset A = \mathbb{C}[x_1,\ldots,x_N]$ supported at $0 \in \mathbb{A}^N$. Such formula involves algebraic data defined through the normalisation morphism

$$\mu_I \colon Z_I \to \mathrm{Bl}_I \mathbb{A}^N.$$

We note here that, when I is monomial, the normalisation Z_I is explicit, being equal to $\mathbb{P}\overline{A[It]}$ (cf. Section 0.10) and one has $Z_I = \mathrm{Bl}_{\overline{I}} \mathbb{A}^2$ when $N=2$ (and I is monomial), where $\overline{I}$ is defined in Equation (0.10.1).

2.7.1 The key example

We present in this subsection the key example (with $N=2$) of the more general formula that will be proven just afterwards (Theorem 2.7.2).

Example 2.7.1. Let $k \geq 2$ be an integer. Then, the ideal $I = (x^k, y^k) \subset \mathbb{C}[x,y]$ satisfies

$$\nu_{\mathbb{C}[x,y]/I} = \ell_{\mathbb{C}[x,y]/I} = k^2,$$

by Example 2.2.4. As explained in Example 0.10.9, the blowup $\mathrm{Bl}_I \mathbb{A}^2$ identifies with the $\mathbb{A}^2$-surface $V(vx^k - uy^k) \subset \mathbb{A}^2 \times \mathbb{P}^1$. As a consequence, $\mathrm{Bl}_I \mathbb{A}^2$ is singular in codimension 1 and hence it is not normal. Then, as observed in Example 0.10.8, Proposition 0.10.2 implies that there is a canonical isomorphism of $\mathbb{A}^2$-schemes between $\mathrm{Bl}_{\mathfrak{m}} \mathbb{A}^2$ and the normalisation of $\mathrm{Bl}_I \mathbb{A}^2$. Under this identification, the normalisation map $\mu_I \colon \mathrm{Bl}_{\mathfrak{m}} \mathbb{A}^2 \to \mathrm{Bl}_I \mathbb{A}^2$ can be realised as the restriction of the morphism

$$\begin{array}{ccc} \mathbb{A}^2 \times \mathbb{P}^1 & \longrightarrow & \mathbb{A}^2 \times \mathbb{P}^1 \\ ((x,y),[u:v]) & \longmapsto & ((x,y),[u^k : v^k]) \end{array}$$

to the subscheme $\mathrm{Bl}_{\mathfrak{m}} \mathbb{A}^2 \subset \mathbb{A}^2 \times \mathbb{P}^1$. The exceptional locus $D = \mathrm{Exc}(\mathrm{Bl}_I \mathbb{A}^2)$ is irreducible and it satisfies

$$\deg\left(\mathrm{Exc}(\mathrm{Bl}_{\mathfrak{m}} \mathbb{A}^2) \xrightarrow{\mu_I} D\right) = k,$$

for such map agrees with the map $\mathbb{P}^1 \to \mathbb{P}^1$ sending $[u:v] \mapsto [u^k : v^k]$. Notice that, if $\varepsilon \colon \mathrm{Bl}_{\mathfrak{m}} \mathbb{A}^2 \to \mathbb{A}^2$ is the blowup map and $Y_I \subset \mathrm{Bl}_{\mathfrak{m}} \mathbb{A}^2$ is the subscheme defined by the ideal sheaf $\varepsilon^{-1}(I) \cdot \mathscr{O}_{\mathrm{Bl}_{\mathfrak{m}} \mathbb{A}^2} \subset \mathscr{O}_{\mathrm{Bl}_{\mathfrak{m}} \mathbb{A}^2}$, then

$$\mathrm{mult}_{\mathrm{Exc}(\mathrm{Bl}_{\mathfrak{m}} \mathbb{A}^2)}(Y_I) = k,$$

hence $\nu_{\mathbb{C}[x,y]/I} = k^2$ is also obtained as

$$k^2 = \deg\Big(\mathrm{Exc}(\mathrm{Bl}_{\mathfrak{m}}\mathbb{A}^2) \xrightarrow{\mu_I} D\Big) \cdot \mathrm{mult}_{\mathrm{Exc}(\mathrm{Bl}_{\mathfrak{m}}\mathbb{A}^2)}(Y_I).$$

2.7.2 The general formula

Let $I \subset A = \mathbb{C}[x_1, \ldots, x_N]$ be the ideal defining a fat point in $\mathbb{A}^N$ supported at $0 \in \mathbb{A}^N$. The normalisation morphism

$$\mu_I : Z_I \to \mathrm{Bl}_I \mathbb{A}^N$$

is a finite morphism by Proposition 0.9.1 (and it is induced by the inclusion of A-algebras $A[It] \hookrightarrow \overline{A[It]}$ in the special case where I is monomial, cf. Section 0.10). Note that Z_I is also a blowup of a fat point in $\mathbb{A}^N$ supported at $0 \in \mathbb{A}^N$, and μ_I is an $\mathbb{A}^N$-morphism. In other words, there is a commutative diagram

$$\begin{tikzcd} Z_I \arrow[rr, "\mu_I"] \arrow[dr, "\bar{\varepsilon}"'] & & \mathrm{Bl}_I \mathbb{A}^N \arrow[dl, "\varepsilon"] \\ & \mathbb{A}^N & \end{tikzcd}$$

where μ_I restricts to a morphism $\mathrm{Exc}(\overline{\varepsilon}) \to \mathrm{Exc}(\varepsilon)$ between exceptional loci. Let

$$D_1, \ldots, D_s \subset E_I \mathbb{A}^N$$

be the irreducible components of the exceptional locus $\mathrm{Exc}(\varepsilon)$, each of which is taken with the reduced structure. Note that, since $E_I\mathbb{A}^N$ is purely of codimension 1, each D_i has dimension $N-1$. For instance, if $N = 2$, each D_i is a (possibly singular) rational curve. Consider the Cartier divisor

$$Y_I = \mu_I^{-1}(E_I\mathbb{A}^N) = \bar{\varepsilon}^{-1}(V(I)) = V(\bar{\varepsilon}^{-1}(I) \cdot \mathscr{O}_{Z_I}) \subset Z_I,$$

and notice that $Y_{I,\mathrm{red}} = \mathrm{Exc}(\bar{\varepsilon})$. Hence Y_I and the exceptional divisor of Z_I share the same irreducible components. For each $i = 1, \ldots, s$, let

$$V_1^{(i)}, \ldots, V_{k_i}^{(i)} \subset Y_I$$

be the irreducible components covering D_i, each taken with the reduced structure. The restrictions

$$\mu_{ij} = \mu_I\big|_{V_j^{(i)}} : V_j^{(i)} \to D_i$$

are finite dominant morphisms of varieties, and we set

$$d_{ij} = \deg \mu_{ij}.$$

The subscheme $Y_I \subset Z_I$ is an effective Cartier divisor, hence it is determined by an invertible ideal sheaf $\mathscr{I} \subset \mathscr{O}_{Z_I}$. Consider the canonical section

$$s_I \in \mathrm{H}^0(Z_I, \mathscr{I}^*) \subset \mathbb{C}(Z_I) = \mathbb{C}(\mathrm{Bl}_I \mathbb{A}^N)$$

attached to the Cartier divisor Y_I. For every pair (i, j), where $i = 1, \ldots, s$ and $j = 1, \ldots, k_i$, we can define

$$e_{ij} = \mathrm{ord}_{V_j^{(i)}}(s_I) = \mathrm{mult}_{V_j^{(i)}}(Y_I),$$

namely we set e_{ij} to be the order of vanishing of the rational function $s_I \in \mathbb{C}(Z_I)$ along the prime $(N-1)$-cycle $V_j^{(i)}$.

We can now state and prove an explicit formula for the Behrend number of I.

Theorem 2.7.2. *Let $I \subset \mathbb{C}[x_1,\dots,x_N]$ be the ideal of a fat point $X \hookrightarrow \mathbb{A}^N$. Then*

$$\nu_{\mathbb{C}[x_1,\dots,x_N]/I} = \sum_{i=1}^{s}\sum_{j=1}^{k_i} d_{ij}e_{ij}.$$

Proof. We know by Proposition 0.9.4 applied to $Y = Z_I$ and $X = \mathrm{Bl}_I \mathbb{A}^N$ that

$$\mathrm{ord}_{D_i}(s_I) = \sum_{j=1}^{k_i} \mathrm{ord}_{V_j^{(i)}}(s_I)\cdot\left[\mathbb{C}(V_j^{(i)}):\mathbb{C}(D_i)\right] = \sum_{j=1}^{k_i} e_{ij}d_{ij}.$$

for every $i = 1,\dots,s$. On the other hand, we have

$$\mathrm{ord}_{D_i}(s_I) = \mathrm{mult}_{D_i}(E_I\mathbb{A}^N).$$

The sought after relation then follows from Equation (2.7.1) by summing over i. □

Example 2.7.3. We generalise here Example 2.7.1. Let $h, k \geq 1$ be integers and let $\delta = \gcd(h,k)$ be their greatest common divisor. Consider the complete intersection ideal $I_{h,k} = (x^h, y^k)$, and the normalisation map

$$\mu\colon \mathrm{Bl}_{\mathfrak{n}_{h,k}} \mathbb{A}^2 \to \mathrm{Bl}_{I_{h,k}} \mathbb{A}^2.$$

Then, the exceptional loci $(E_{\mathfrak{n}_{h,k}}\mathbb{A}^2)_{\mathrm{red}}$ and $(E_{I_{h,k}}\mathbb{A}^2)_{\mathrm{red}}$ are both irreducible, and $\deg \mu|_{(E_{\mathfrak{n}_{h,k}}\mathbb{A}^2)_{\mathrm{red}}} = \delta$. This follows from the formulas in Lemma 2.6.2 and Theorem 2.7.2, namely

$$\mathfrak{n}_{h,k} = \mathfrak{n}_{h',k'}^{\delta}$$

where $k' = k/\delta$ and $h' = h/\delta$.

For instance, given $I_{4,6} = (x^4, y^6)$, one has $\mathfrak{n}_{4,6} = \mathfrak{n}_{2,3}^2$ and, as a consequence of Proposition 2.2.8, up to isomorphism, the normalisation map has the form

$$\mu\colon \mathrm{Bl}_{\mathfrak{n}_{2,3}} \mathbb{A}^2 \to \mathrm{Bl}_{I_{4,6}} \mathbb{A}^2.$$

Then, with the same notation as in Theorem 2.7.2, $\nu_{\mathbb{C}[x,y]/I_{4,6}} = 24$ because $I_{4,6}$ is a complete intersection, $d = \deg(\mu|_{E_{\mathfrak{n}_{2,3}}}) = 2 = \gcd(4,6)$ because of what we just said, and a direct computation in toric geometry (see Section 2.5.1) shows $e = 12$ where, if $\varepsilon\colon \mathrm{Bl}_{\mathfrak{n}_{2,3}}\mathbb{A}^2 \to \mathbb{A}^2$ denotes the blowup map, then $e = \mathrm{mult}_{E_{\mathfrak{n}_{2,3}}}(\varepsilon^{-1}(I_{4,6})\cdot\mathscr{O}_{\mathrm{Bl}_{\mathfrak{n}_{2,3}}\mathbb{A}^2})$.

Example 2.7.4. Let $I \subset \mathfrak{m} \subset \mathbb{C}[x,y]$ be an ideal of finite colength generated by $s+1$ monomials

$$m_0,\dots,m_s \in \mathbb{C}[x,y],$$

all of degree δ. Then, by Proposition 0.10.2, the normalisation of $\mathrm{Bl}_I \mathbb{A}^2$ is given by $\mathrm{Bl}_{\mathfrak{m}}\mathbb{A}^2$. In particular, the exceptional locus $\mathrm{Exc}(\mathrm{Bl}_I \mathbb{A}^2)$ is irreducible. Consider the rational map

$$\begin{aligned} \mathbb{A}^2 &\overset{\varphi_I}{\dashrightarrow} \mathbb{P}^s \\ (a,b) &\longmapsto [m_0(a,b):\cdots:m_s(a,b)] \end{aligned}$$

whose indeterminacy locus is the origin $\{0\} = V(\sqrt{I}) \subset \mathbb{A}^2$. The fact that all the monomials have the same degree δ implies that φ_I induces a morphism

$$\begin{aligned} \mathbb{P}^1 &\xrightarrow{\quad\overline{\varphi}_I\quad} \mathbb{P}^s \\ [a:b] &\longmapsto [m_0(a,b):\cdots:m_s(a,b)]. \end{aligned}$$

Let $X \subset \mathbb{A}^2 \times \mathbb{P}^s$ be the Zariski closure of the graph of the map φ_I, i.e.

$$X = \overline{\left\{((a,b),q) \in (\mathbb{A}^2 \smallsetminus \{0\}) \times \mathbb{P}^s \;\middle|\; q = [m_0(a,b):\cdots:m_s(a,b)]\right\}} \subset \mathbb{A}^2 \times \mathbb{P}^s.$$

Then, by Proposition 0.2.2, there is a canonical isomorphism of $\mathbb{A}^2$-schemes $\mathrm{Bl}_I \mathbb{A}^2 \cong X$. We claim that, if we identify $\mathbb{P}^s \cong \{0\} \times \mathbb{P}^s \subset \mathbb{A}^2 \times \mathbb{P}^s$, then $\mathrm{Im}(\overline{\varphi}_I) = \mathrm{Exc}(\varepsilon_I)$ where $\varepsilon_I : X \to \mathbb{A}^2$ is the structure morphism, i.e. the restricion of the canonical projection onto $\mathbb{A}^2$. Since $\mathrm{Im}(\overline{\varphi}_I)$ has dimension 1 and $\mathrm{Exc}(\varepsilon_I)$ is irreducible, in order to prove $\mathrm{Im}(\overline{\varphi}_I) = \mathrm{Exc}(\varepsilon_I)$, it is enough to prove $\mathrm{Im}(\overline{\varphi}_I) \subset \mathrm{Exc}(\varepsilon_I)$.

Let $p = (0, [m_0(a,b):\cdots:m_s(a,b)])$ be a point in $\mathrm{Im}(\overline{\varphi}_I)$ and let $L_{a,b} = V(bx - ay) \subset \mathbb{A}^2$ be a line trough the origin of $\mathbb{A}^2$. Let us denote by $\varphi_{a,b}$ the restriction $\varphi_{a,b} = \varphi_I|_{L_{a,b}}$ and by $X_{a,b} \subset X$ the Zariski closure of its graph in $\mathbb{A}^2 \times \mathbb{P}^s$. Then, the map $\varphi_{a,b}$ is constant and we have

$$\varphi_{a,b} \equiv [m_0(a,b):\cdots:m_s(a,b)].$$

As a consequence, $p \in X_{a,b} \subset X$ which proves $\mathrm{Im}(\overline{\varphi}_I) \subset \mathrm{Exc}(\varepsilon_I)$.

Notice that, if $k > 0$ is an integer and $J \subset \mathbb{C}[x,y]$ is the ideal $J = (m_0^k, \dots, m_s^k)$, then $\mathrm{Im}(\overline{\varphi}_I) = \mathrm{Im}(\overline{\varphi}_J)$ and

$$\deg(\overline{\varphi}_J : \mathbb{P}^1 \to \mathrm{Im}(\overline{\varphi}_J)) = k \cdot \deg(\overline{\varphi}_I : \mathbb{P}^1 \to \mathrm{Im}(\overline{\varphi}_I)).$$

Proposition 2.7.5. *Let $h, k \in \mathbb{N}$ be two positive integers and let $\delta = \gcd(h,k)$ be their greatest common divisor. Consider the ideals $I = (x^h, y^h)$, $J = (x^k, y^k)$ in $\mathbb{C}[x,y]$, and their product*

$$IJ = (x^{h+k}, x^k y^h, x^h y^k, y^{h+k}).$$

Then, the Behrend number of the subscheme defined by $IJ \subset \mathbb{C}[x,y]$ is

$$\nu_{\mathbb{C}[x,y]/IJ} = \delta \cdot (h+k).$$

Proof. First of all we observe that, by Proposition 0.10.2, there is a canonical isomorphism of $\mathbb{A}^2$-schemes between $\mathrm{Bl}_{\mathfrak{m}} \mathbb{A}^2$ and the normalisation of $\mathrm{Bl}_{IJ} \mathbb{A}^2$. We have (see Section 0.9) the following commutative diagram

$$\begin{tikzcd}
& \mathrm{Bl}_I \mathbb{A}^2 \arrow[dr, bend left, "\varepsilon_I"] & \\
\mathrm{Bl}_{\mathfrak{m}} \mathbb{A}^2 \arrow[ur, bend left, "\mu_I"] \arrow[r, "\mu_{IJ}"] \arrow[dr, bend right, "\mu_J"'] & \mathrm{Bl}_{IJ} \mathbb{A}^2 \arrow[u, "\theta_I"'] \arrow[r, "\varepsilon_{IJ}"] \arrow[d, "\theta_J"] & \mathbb{A}^2 \\
& \mathrm{Bl}_J \mathbb{A}^2 \arrow[ur, bend right, "\varepsilon_J"'] &
\end{tikzcd}$$

where all the maps are birational morphisms. The maps $\varepsilon_I, \varepsilon_J, \varepsilon_{IJ}$ and θ_I, θ_J are the blowup morphisms and μ_I, μ_J, μ_{IJ} are the normalisation morphisms. Moreover, any composition $\mathrm{Bl}_{\mathfrak{m}} \mathbb{A}^2 \to \mathbb{A}^2$ which connects $\mathrm{Bl}_{\mathfrak{m}} \mathbb{A}^2$ and $\mathbb{A}^2$ coincides with the blowup map $\varepsilon_{\mathfrak{m}} \colon \mathrm{Bl}_{\mathfrak{m}} \mathbb{A}^2 \to \mathbb{A}^2$. Notice that, since $\mathrm{Exc}(\varepsilon_{\mathfrak{m}})$ is irreducible, also $\mathrm{Exc}(\varepsilon_I), \mathrm{Exc}(\varepsilon_J)$ and $\mathrm{Exc}(\varepsilon_{IJ})$ are irreducible because they are dominated by $\mathrm{Exc}(\varepsilon_{\mathfrak{m}})$.

As a consequence, in order to compute (through Theorem 2.7.2) the Behrend number of the ideal IJ, we have to compute only two numbers, namely

$$e = \mathrm{mult}_{\mathrm{Exc}(\varepsilon_{\mathfrak{m}})}\left(\varepsilon_{\mathfrak{m}}^{-1}(IJ) \cdot \mathscr{O}_{\mathrm{Bl}_{\mathfrak{m}} \mathbb{A}^2}\right)$$

and

$$d = \deg\left(\mu_{IJ}\big|_{\mathrm{Exc}(\varepsilon_{\mathfrak{m}})}\right).$$

We start from the computation of e. As usual, $\mathrm{Bl}_{\mathfrak{m}} \mathbb{A}^2$ is covered by two charts U_0 and U_1 isomorphic to $\mathbb{A}^2$. In order to compute e, it is enough to focus on U_0. We introduce toric coordinates a, b and the map $\varepsilon_{\mathfrak{m}}$ restricts to the map

$$\begin{aligned} U_0 &\xrightarrow{\varepsilon_{\mathfrak{m}}|_{U_0}} \mathbb{A}^2 \\ (a,b) &\longmapsto (ab, b). \end{aligned}$$

Hence, we have

$$\varepsilon_{\mathfrak{m}}\big|_{U_0}^{-1}(IJ) \cdot \mathbb{C}[a,b] = (b^{h+k}) \subset \mathbb{C}[a,b],$$

and, as a consequence, $e = h + k$.

Now we move to the computation of d. We split this computation in two steps.

Step 1: Suppose $\delta = 1$.

Since all the exceptional loci of the varieties in the above diagram are irreducible rational curves and the involved maps are all dominant, we get a commutative diagram of fields extensions

$$\begin{tikzcd}
 & \mathbb{C}(t) \arrow[dl, hook', "\varphi_I"'] \arrow[d, hook, "\psi_I"] \\
\mathbb{C}(t) & \mathbb{C}(t) \arrow[l, hook', "\varphi_{IJ}"'] \\
 & \mathbb{C}(t) \arrow[ul, hook, "\varphi_J"] \arrow[u, hook', "\psi_J"']
\end{tikzcd}$$

where, up to canonical identifications, we have

$$\varphi_\bullet = \mu_\bullet\big|^*_{\mathrm{Exc}(\mathrm{Bl}_{\mathfrak{m}} \mathbb{A}^2)} \colon \mathbb{C}(\mathrm{Exc}(\mathrm{Bl}_\bullet \mathbb{A}^2)) \longrightarrow \mathbb{C}(\mathrm{Exc}(\mathrm{Bl}_{\mathfrak{m}} \mathbb{A}^2))$$

and

$$\psi_\bullet = \theta_\bullet\big|^*_{\mathrm{Exc}(\mathrm{Bl}_{IJ} \mathbb{A}^2)} \colon \mathbb{C}(\mathrm{Exc}(\mathrm{Bl}_\bullet \mathbb{A}^2)) \longrightarrow \mathbb{C}(\mathrm{Exc}(\mathrm{Bl}_{IJ} \mathbb{A}^2)).$$

Now, as a consequence of general field theory and of Example 2.7.1, we have

$$\begin{gathered}[\mathbb{C}(t):\psi_I(\mathbb{C}(t))]\cdot[\mathbb{C}(t):\varphi_{IJ}(\mathbb{C}(t))]=[\mathbb{C}(t):\varphi_I(\mathbb{C}(t))]\\ [\mathbb{C}(t):\psi_J(\mathbb{C}(t))]\cdot[\mathbb{C}(t):\varphi_{IJ}(\mathbb{C}(t))]=[\mathbb{C}(t):\varphi_J(\mathbb{C}(t))]\\ [\mathbb{C}(t):\varphi_I(\mathbb{C}(t))]=h\\ [\mathbb{C}(t):\varphi_J(\mathbb{C}(t))]=k,\end{gathered}$$

which, together with the hypothesis $\delta=\gcd(h,k)=1$, imply

$$\begin{gathered}[\mathbb{C}(t):\psi_I(\mathbb{C}(t))]=h,\\ [\mathbb{C}(t):\psi_J(\mathbb{C}(t))]=k,\\ [\mathbb{C}(t):\varphi_{IJ}(\mathbb{C}(t))]=1.\end{gathered}$$

Thus, $d=\delta=1$.

Step 2: Suppose $\delta>1$. Consider the positive integers $h'=h/\delta$ and $k'=k/\delta$ and the ideals $I'=(x^{h'},y^{h'})$ and $J'=(x^{k'},y^{k'})$. Let f, f' and g be the rational maps defined as follows:

$$\begin{aligned}\mathbb{A}^2 &\overset{f}{\dashrightarrow} \mathbb{P}^3\\ (x,y) &\longmapsto [x^{h+k}:x^hy^k:x^ky^h:y^{h+k}]\end{aligned}$$

$$\begin{aligned}\mathbb{A}^2 &\overset{f'}{\dashrightarrow} \mathbb{P}^3\\ (x,y) &\longmapsto [x^{h'+k'}:x^{h'}y^{k'}:x^{k'}y^{h'}:y^{h'+k'}]\end{aligned}$$

$$\begin{aligned}\mathbb{P}^3 &\xrightarrow{\quad g\quad} \mathbb{P}^3\\ [w_0:w_1:w_2:w_3] &\longmapsto [w_0^\delta:w_1^\delta:w_2^\delta:w_3^\delta].\end{aligned}$$

Then, a trivial computation shows that the diagram

$$\begin{tikzcd} \mathbb{A}^2 \arrow[r, dashed, "f'"] \arrow[d, dashed, "f"'] & \mathbb{P}^3 \arrow[dl, "g"] \\ \mathbb{P}^3 & \end{tikzcd}$$

commutes. By Proposition 0.2.2, we have canonical isomorphisms of $\mathbb{A}^2$-schemes

$$X\cong \mathrm{Bl}_{IJ}\mathbb{A}^2,\qquad X'\cong \mathrm{Bl}_{I'J'}\mathbb{A}^2 \tag{2.7.2}$$

where, if we denote by $\Gamma(f)$ and $\Gamma(f')$ the graphs of the rational maps f and f', then $X,X'\subset \mathbb{A}^2\times\mathbb{P}^3$ are respectively defined as the Zariski closures of $\Gamma(f)$ and $\Gamma(f')$, i.e. $X=\overline{\Gamma(f)}$ and $X'=\overline{\Gamma(f')}$, and the $\mathbb{A}^2$-structure morphism is given, in both cases, by the restriction of the first projection. Define now the morphism λ_{IJ} as the restriction of the map

$$\mathrm{id}_{\mathbb{A}^2}\times g:\mathbb{A}^2\times\mathbb{P}^3\to\mathbb{A}^2\times\mathbb{P}^3$$

to X'. Up to the identifications (2.7.2) we have a commutative diagram

$$\begin{tikzcd}
 & \mathrm{Bl}_{I'}\mathbb{A}^2 \arrow[r, "\lambda_I"] & \mathrm{Bl}_I\mathbb{A}^2 \arrow[dr, bend left, "\varepsilon_I"] & \\
\mathrm{Bl}_{\mathrm{m}}\mathbb{A}^2 \arrow[ur, bend left, "\mu_{I'}"] \arrow[r, "\mu_{I'J'}"] \arrow[dr, bend right, "\mu_{J'}"'] & \mathrm{Bl}_{I'J'}\mathbb{A}^2 \arrow[u, "\theta_{I'}"'] \arrow[d, "\theta_{J'}"] \arrow[r, "\lambda_{IJ}"] & \mathrm{Bl}_{IJ}\mathbb{A}^2 \arrow[u, "\theta_I"'] \arrow[d, "\theta_J"] \arrow[r, "\varepsilon_{IJ}"] & \mathbb{A}^2 \\
 & \mathrm{Bl}_{J'}\mathbb{A}^2 \arrow[r, "\lambda_J"] & \mathrm{Bl}_J\mathbb{A}^2 \arrow[ur, bend right, "\varepsilon_J"'] &
\end{tikzcd}$$

where

- the maps $\mu_{I'}, \mu_{J'}$ and $\mu_{I'J'}$ are the normalisation morphisms,
- the maps λ_I, λ_J are defined similarly to λ_{IJ},
- the compositions $\mu_I = \lambda_I \circ \mu_{I'}, \mu_J = \lambda_J \circ \mu_{J'}$ and $\mu_{IJ} = \lambda_{IJ} \circ \mu_{I'J'}$ are the normalisation morphisms mentioned above,
- any composition $\mathrm{Bl}_\bullet \mathbb{A}^2 \to \mathbb{A}^2$ agrees with the blowup map $\varepsilon_\bullet \colon \mathrm{Bl}_\bullet \mathbb{A}^2 \to \mathbb{A}^2$.

We know, by general theory, that

$$\deg\left(\mu_{IJ}\big|_{\mathrm{Exc}(\varepsilon_{\mathrm{m}})}\right) = \deg\left(\mu_{I'J'}\big|_{\mathrm{Exc}(\varepsilon_{\mathrm{m}})}\right) \cdot \deg\left(\lambda_{IJ}\big|_{\mathrm{Exc}(\varepsilon_{I'J'})}\right)$$

and we also know, by **Step 1**, that

$$\deg\left(\mu_{I'J'}\big|_{\mathrm{Exc}(\varepsilon_{\mathrm{m}})}\right) = 1.$$

Therefore, we have

$$d = \deg\left(\lambda_{IJ}\big|_{\mathrm{Exc}(\varepsilon_{I'J'})}\right).$$

Finally, as a consequence of Example 2.7.4, if we call $E = (X' \cap \{0\} \times \mathbb{P}^3)_{\mathrm{red}}$ the exceptional locus of X', then we have

$$\deg \lambda_{IJ}\big|_E = \delta.$$

This complete the proof. $\square$

Example 2.7.6. For $h = k$ we find the formula

$$\nu_{\mathbb{C}[x,y]/(x^h,y^h)^2} = 2h^2 = 2 \cdot \nu_{\mathbb{C}[x,y]/(x^h,y^h)},$$

which may also be deduced from Proposition 2.2.8. While, for $I = (x^2, y^2)(x^3, y^3)$ and $J = (x^2, y^2)(x^6, y^6)$ we find

$$\nu_{\mathbb{C}[x,y]/I} = 5, \qquad \nu_{\mathbb{C}[x,y]/J} = 16.$$

Remark 2.7.7. Let $\mu_\bullet$, for $\bullet \in \{I, J, IJ, I', J', I'J'\}$, be defined as in the proof of Proposition 2.7.5, and let $\vartheta_\bullet$ be the restrictions $\vartheta_\bullet = \mu_\bullet|_{\mathrm{Exc}(\mathrm{Bl}_{\mathrm{m}} \mathbb{A}^2)}$. Then, up to isomorphism, the maps $\vartheta_\bullet$ are of the form

$$\vartheta_\bullet = \pi \circ \mathrm{v}_{1,d} : \mathbb{P}^1 \to \mathbb{P}^s$$

where

$$\mathsf{v}_{1,d} : \mathbb{P}^1 \to \mathbb{P}^d$$

is the d-th Veronese embedding of $\mathbb{P}^1$ for some positive integer $d \geq 1$ and

$$\pi : \mathbb{P}^d \dashrightarrow \mathbb{P}^s$$

is the projection onto some coordinate projective subspace of dimension $s \leq d$.

The blowup $\mathrm{Bl}_{IJ}\,\mathbb{A}^2$ of a product of ideals as in Proposition 2.7.5 has a peculiarity that we find here for the first time: its exceptional locus is (in general) not normal. For example, if $I = (x^3, y^3)$ and $J = (x^2, y^2)$, the exceptional locus $\mathrm{Exc}(\mathrm{Bl}_{IJ}\,\mathbb{A}^2)$ has two cusps as singularities. The general situation is described by the following result.

Proposition 2.7.8. *Fix positive integers $h, k > 1$ and let $\delta = \gcd(h,k)$ be their greatest common divisor. Set*

$$I = (x^h, y^h)\cdot(x^k, y^k) = (x^{h+k}, x^h y^k, x^k y^h, y^{h+k}) \subset \mathbb{C}[x,y].$$

Then, the exceptional locus of $\mathrm{Bl}_I\,\mathbb{A}^2$ is an irreducible projective rational curve with two singularities of local equations of the form $\alpha^{h/\delta} - \beta^{k/\delta} = 0$.

Proof. As explained in Example 2.7.4, the image of the map

$$\begin{aligned} \mathbb{P}^1 &\xrightarrow{\quad\varphi_I\quad} \mathbb{P}^3 \\ [a:b] &\longmapsto [a^{h+k} : a^h b^k : a^k b^h : b^{h+k}] \end{aligned}$$

is isomorphic to the exceptional locus of $\mathrm{Bl}_I\,\mathbb{A}^2$. The statement follows now by an easy computation. □

Remark 2.7.9. Let $h, k > 1$ be two natural numbers such that $\gcd(h,k) = 1$ and $h^2 < h + k < k^2$. Consider the ideals $I = (x^h, y^h)$ and $J = (x^k, y^k)$. Then, we have the following inequalities:

$$\nu_{\mathbb{C}[x,y]/I} < \nu_{\mathbb{C}[x,y]/IJ} < \nu_{\mathbb{C}[x,y]/J}.$$

For instance, this happens for $h = 2$ and $k = 3$.

Corollary 2.7.10 (of Proposition 2.7.5)**.** *Let $d_1, \ldots, d_s$ be positive integers. Given the ideals $I_k = (x^{d_k}, y^{d_k})$, for $k = 1, \ldots, s$, we have*

$$\nu_{\mathbb{C}[x,y]/I_1 \cdots I_s} = \gcd(d_1, \ldots, d_s) \cdot \sum_{k=1}^{s} d_k.$$

Example 2.7.11. Consider the ideal $I = (x,y)\cdot(x^2,y^2)\cdot(x^3,y^3)\cdots(x^s,y^s)$ then

$$\nu_{\mathbb{C}[x,y]/I} = \sum_{k=1}^{s} k = \binom{s+1}{2}.$$

2.8 Some difficulties in dimension 3

In Section 2.5 we proved that any normal monomial ideal $I \subset \mathbb{C}[x,y]$ factors in a unique way as a product of powers of ideals of the form $\mathfrak{n}_{\alpha,\beta}$. Furthermore, we noticed in Theorem 2.6.12 that there is a bijective correspondence between the ideals that appear in such factorisation and the irreducible components of the exceptional divisor of the blowup $\mathrm{Bl}_I \mathbb{A}^2$. This correspondence has allowed us, in numerous cases, to calculate the Behrend number of I. Unfortunately, as we show in the discusion below, the situation is more complicated in higher dimension.

Let $I, J \subset \mathbb{C}[x,y,z]$ be the curvilinear ideals defined by

$$I = (x^2, y, z), \qquad J = (x, y^2, z),$$

and let $\mathfrak{m}_{\mathbb{A}^3} = (x,y,z) \subset \mathbb{C}[x,y,z]$, be the maximal ideal of the origin $0 \in \mathbb{A}^3$. We want to study the blowup of the ideal IJ. We will show that the exceptional divisor $E_{IJ}\mathbb{A}^3$ decomposes into three irreducible components instead of the expected two.

First, we deal with the blowup $\varepsilon_I \colon B_I = \mathrm{Bl}_I \mathbb{A}^3 \to \mathbb{A}^3$ and then we will move to the analysis of $\mathrm{Bl}_{IJ} \mathbb{A}^3$. Since I is a complete intersection, we have

$$\nu_{\mathbb{C}[x,y,z]/I} = \ell_{\mathbb{C}[x,y,z]/I} = 2,$$

by Example 2.2.4. Moreover, as a consequence of [21, Ex. IV-26], we have

$$B_I = \left\{ ((x,y,z),[u_0:u_1:u_2]) \,\middle|\, \mathrm{rank}\begin{pmatrix} x & y & z^2 \\ u_0 & u_1 & u_2 \end{pmatrix} \le 1 \right\} \subset \mathbb{A}^3 \times \mathbb{P}^2.$$

Notice that $\mathrm{Exc}(\varepsilon_I) \cong \mathbb{P}^2$ and that the threefold B_I is singular along the projective line

$$L = \left\{ ((0,0,0),[\lambda:\mu:0]) \,\middle|\, [\lambda:\mu] \in \mathbb{P}^1 \right\} \subset B_I.$$

Let us now focus on the open neighbourhood of L defined by

$$U = B_I \cap ((\mathbb{A}^3 \times \{u_0 \neq 0\}) \cup (\mathbb{A}^3 \times \{u_1 \neq 0\})).$$

The projection $U \to \mathbb{P}^1$ sending $(p,[u_0:u_1:u_2]) \mapsto [u_0:u_1]$ is an isotrivial family of singular surfaces of type A_1, which shows that B_I is normal (this can also be deduced from a general version of Proposition 0.10.2, see [17, Prop. 1.1]). Notice that the base of the family corresponds to the pencil of planes containing $V(I)$ as a closed subscheme.

Alternatively, similarly as we have done in Example 2.3.10, we could have built B_I in the following way. Let $\varepsilon_{\mathfrak{m}_{\mathbb{A}^3}} \colon B_{\mathfrak{m}_{\mathbb{A}^3}} = \mathrm{Bl}_{\mathfrak{m}_{\mathbb{A}^3}} \mathbb{A}^3 \to \mathbb{A}^3$ be the blowup of $\mathbb{A}^3$ at the origin. Then, a direct computation shows that

$$\varepsilon_{\mathfrak{m}_{\mathbb{A}^3}}^{-1}(I) \cdot \mathscr{O}_{B_{\mathfrak{m}_{\mathbb{A}^3}}} = \mathscr{H}_1 \cdot \mathscr{H}_2,$$

where $\mathscr{H}_1$ is the ideal sheaf of a Cartier divisor and $\mathscr{H}_2$ is the ideal sheaf of a reduced point $p \in \mathrm{Exc}(\varepsilon_{\mathfrak{m}_{\mathbb{A}^3}}) \subset B_{\mathfrak{m}_{\mathbb{A}^3}}$. Therefore, the decomposition into irreducible components of the exceptional locus of $B_{\mathfrak{m}_{\mathbb{A}^3}\cdot I} = \mathrm{Bl}_{\mathfrak{m}_{\mathbb{A}^3}\cdot I} \mathbb{A}^3$ is given by

$$\mathrm{Exc}(B_{\mathfrak{m}_{\mathbb{A}^3}\cdot I}) = S_1 \cup S_2,$$

where

- $S_1 \cong \mathbb{P}^2$,
- $S_2 \cong \mathrm{Bl}_q \mathbb{P}^2$ for some $q \in \mathbb{P}^2$, and agrees with the strict transform of the exceptional locus $\mathrm{Exc}(B_{\mathfrak{m}_{\mathbb{A}^3}})$ via the blowup map $\lambda_{\mathfrak{m}_{\mathbb{A}^3}} : B_{\mathfrak{m}_{\mathbb{A}^3}\cdot I} \to B_{\mathfrak{m}_{\mathbb{A}^3}}$ induced by Lemma 0.2.1,
- $S_1 \cap S_2 = \mathrm{Exc}(S_2) \cong \mathbb{P}^1$.

Now, consider the following canonical morphisms

$$B_{\mathfrak{m}_{\mathbb{A}^3}} \xleftarrow{\lambda_{\mathfrak{m}_{\mathbb{A}^3}}} B_{\mathfrak{m}_{\mathbb{A}^3}\cdot I} \xrightarrow{\lambda_I} B_I$$

where the existence of λ_I follows by the universal property of B_I. Since $B_{\mathfrak{m}_{\mathbb{A}^3}\cdot I}$ and $B_{\mathfrak{m}_{\mathbb{A}^3}}$ are smooth and B_I is normal, all the morphisms above have connected fibres by Theorem 0.9.2. In particular, they are isomorphisms outside their exceptional loci, $\lambda_I|_{S_1} : S_1 \to \mathrm{Exc}(B_I)$ is an isomorphism and $\lambda_I|_{S_2} : S_2 \to L \subset \mathrm{Exc}(B_I)$ coincides with the tautological projection of the blowup of the projective plane at a point.

Both constructions confirm the correspondence in Theorem 2.6.12. Unfortunately, such relation cannot, in general, be expected in dimension 3. To see this, consider this time the ideal $K = IJ$. Then, as above, we have canonical morphisms

$$B_{\mathfrak{m}_{\mathbb{A}^3}} \xleftarrow{\theta_{\mathfrak{m}_{\mathbb{A}^3}}} B_{\mathfrak{m}_{\mathbb{A}^3}\cdot K} \xrightarrow{\theta_K} B_K$$

and we can apply again Theorem 0.9.2 because $B_{\mathfrak{m}_{\mathbb{A}^3}\cdot K}$ and $B_{\mathfrak{m}_{\mathbb{A}^3}}$ are smooth and B_K is normal as per Proposition 0.10.2. The analogue of the description above is

- $\mathrm{Exc}(B_{\mathfrak{m}_{\mathbb{A}^3}\cdot K}) = S_1 \cup S_2 \cup S_3$,
- $S_2 \cong \mathbb{P}^2 \cong S_3$,
- $S_1 \cong \mathrm{Bl}_{q_1,q_2} \mathbb{P}^2$, and agrees with the strict transform of the exceptional locus $\mathrm{Exc}(B_{\mathfrak{m}_{\mathbb{A}^3}})$ via the blowup map $\theta_{\mathfrak{m}_{\mathbb{A}^3}}$,
- $S_2 \cap S_3 = \emptyset$,
- $S_i \cap S_1 = L_i$ for $i = 2,3$, where L_2 and L_3 are the irreducible (disjoint) components of $\mathrm{Exc}(\theta_{\mathfrak{m}_{\mathbb{A}^3}}|_{S_1})$.

Finally, one can prove that the map θ_K contracts one line to a singular point, namely the strict transform (via $\theta_{\mathfrak{m}_{\mathbb{A}^3}}$) of the line trough the two points that correspond to q_1, q_2 via the isomorphism $S_1 \cong \mathrm{Bl}_{q_1,q_2} \mathbb{P}^2$ mentioned above. As a consequence, the irreducible components of $\mathrm{Exc}(B_K)$ are:

$$\theta_K(S_1) \cong \mathbb{P}^1 \times \mathbb{P}^1, \quad \theta_K(S_2) \cong \mathbb{P}^2 \cong \theta_K(S_3).$$

One can also find, via a direct computation, the Behrend number of the ideal K, which is

$$\nu_{\mathbb{C}[x,y,z]/K} = 8.$$

The above discussion shows that, even for towers, generalising to dimension 3 the constructions and algorithms carried out in Sections 2.3 and 2.4 is a nontrivial task, that we leave for future research.

Appendix A

A tale of blowups

The blowup of an n-dimensional quasi-projective smooth variety X along a (closed) k-dimensional smooth submanifold Z is well understood in all dimensions (see, for instance [21, Chapter 4]). This is a consequence of the fact that the blowup can be done in analytic coordinates. More precisely, one can write $X = U \cup V$ where U, V are open in the euclidean topology such that

- $V \cap Z = \emptyset$, and U is an analytical neighbourhood of the submanifold Z,
- $\forall p \in Z$ there exists an analytic neighbourhood W of p in X such that the pair $(Z \cap U \cap W, U \cap W)$ is (analytically) isomorphic to a pair $(\mathbb{C}^k \times \{0_{\mathbb{C}^{n-k}}\}, \mathbb{C}^k \times \mathbb{C}^{n-k})$,

now one can apply [21, Prop. IV-25] to compute $\mathrm{Bl}_{Z\cap U\cap W} U \cap W$, and glue all the opens so finding $\mathrm{Bl}_Z X$. This procedure is well posed as proven, for instance, in [33, Sec 4.6.2].

Quite less clear, however, is the blowup along singular subvarieties. The first non-trivial case is that of fat points on surfaces that we discussed in Chapter 2, the second is that of fat points and singular curves in dimension 3. Up to analytical equivalence, the curve singularities that one can have are combinations of the following three kinds:

- non reduced components,
- embedded fat points,
- (isolated) singularities of integral curves.

In this appendix we will discuss the simplest case, namely 1-dimensional nodes which, analytically, can be treated as 2 incident lines.

When we talk about blowup with centre the union of two incident lines $L, M \subset \mathbb{A}^3$, we mean the blowup with the centre the ideal intersection $I_L \cap I_M$, where I_L and I_M are respectively the ideals of L and M. Despite this, one can carry out several processes and obtain different varieties. For instance, one can first blowup one line and then the other or, one can first separate the lines by blowing up the origin and then blowup the strict transform of the lines or even blowup directly the product ideal. All these choices produce different birational models of the affine space which interconnected as described in the Figure A.1.

Notice that the blowups along intersection ideal and product ideal produce varieties with an isolated (singular) conifold point. The resolutions of these conifolds are also shown in Figure A.1.

LEGEND			
	Projective line $\mathbb{P}^1$		Affine line $\mathbb{A}^1$
	0-th Hirzebruch $\mathbb{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$		$\mathbb{P}^1 \times \mathbb{C}$
	Projective plane $\mathbb{P}^2$		$\mathrm{Bl}_{p_1}(\mathbb{P}^1 \times \mathbb{C})$
	degree 7 Del Pezzo $\mathrm{dP}_7 \cong \mathrm{Bl}_{p_1,p_2}\mathbb{P}^2$	$\rightarrow$	Small resolution
← - →	Flop	○	Conifold point

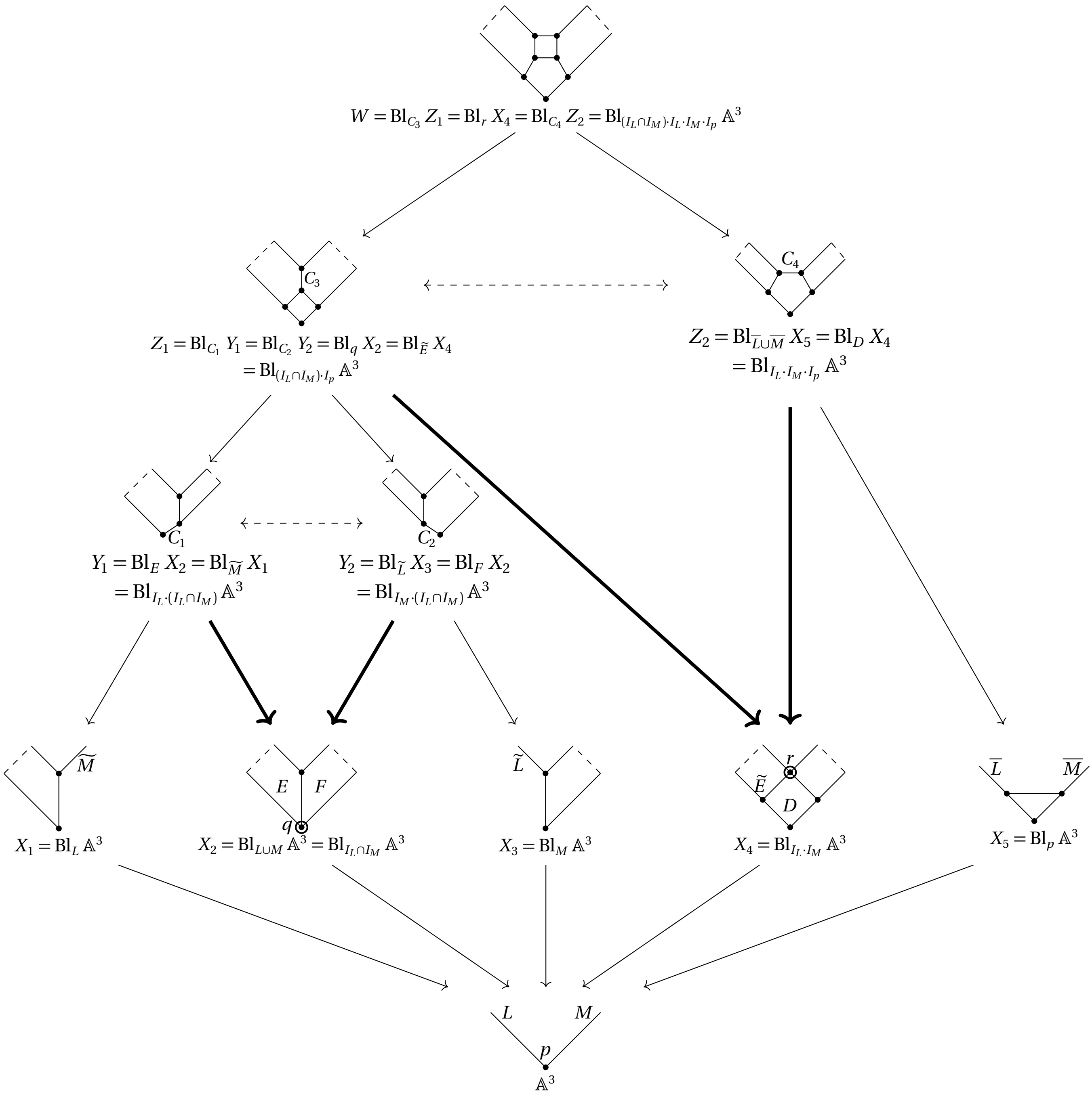


Figure A.1. An example of blowups of $\mathbb{A}^3$

Appendix B

Inclusion–exclusion principle in algebraic geometry

We conclude the thesis with an appendix of a more philosophical than technical nature. Also in this appendix we deal with blowups at reducible and non-reduced centres.

In what follows, we will denote by V the usual functor that associates, to each ideal I of a ring R, the following subscheme of $\operatorname{Spec} A$:

$$V(I) = \operatorname{Spec}\left(\frac{A}{I}\right).$$

It is remarkable that even if the union of two subschemes $Z_I = V(I)$ and $Z_J = V(J)$, of a scheme $\operatorname{Spec}(A)$, is, by definition, $V(I \cap J)$, also the subscheme $V(I \cdot J)$, in some sense, can be thought as a union. A union with multiplicity! The following example should serve as a clarification.

Example B.0.1. Let $I = (x)$ and $J = (x, y)$ be two ideals of $\mathbb{C}[x, y]$. They define the y-axis $V(I)$ and the origin $V(J)$ of $\mathbb{A}^2 = \operatorname{Spec}\mathbb{C}[x, y]$. We have $I \subset J$ and, as a consequence, $V(J) \subset V(I)$. Therefore,

$$V(I) = V(I \cap J) = V(I) \cup V(J).$$

On the other hand, $I \cdot J = (x^2, xy)$ is the ideal of the y-axis with an embedded origin added.

Let $I_1, I_2 \subset \mathbb{C}[x_1, \dots, x_n]$ be two ideals and let $Z_1 = V(I_1)$ and $Z_2 = V(I_2)$ the corresponding subschemes of the affine space $\mathbb{A}^n$. We want to understand the blowup $\operatorname{Bl}_{Z_1 \cup Z_2} \mathbb{A}^n = \operatorname{Bl}_{I_1 \cap I_2} \mathbb{A}^n$ and we want to compare it with $\operatorname{Bl}_{I_1 \cdot I_2} \mathbb{A}^n$. This type of comparison is very useful in practice when one wants to make explicit calculations. For instance, the blowup $\operatorname{Bl}_{I_1 \cdot I_2} \mathbb{A}^n$ is better understood, via Lemma 0.2.1, than $\operatorname{Bl}_{I_1 \cap I_2} \mathbb{A}^n$.

We would like to have a decomposition of the following form:

$$\text{“}Z_1 \cup Z_2 = ((Z_1 \cup Z_2) \smallsetminus (Z_1 \cap Z_2)) \cup (Z_1 \cap Z_2)\text{”} \tag{B.0.1}$$

which would help break down the problem into simpler subproblems. Although it seems that the Zariski topology is so rigid that it does not allow such a decomposition, a formula similar to (B.0.1) can be obtained thanks to the following lemma.

Lemma B.0.2. *Let R be a ring and let $I, J \subset R$ be two ideals. Then,*

$$I \cap J = ((I \cdot J) : (I + J)).$$

Remark B.0.3. One can show that the division operation between ideals corresponds to the following operation between (reduced) schemes:

$$V(I : J) = \overline{V(I) \smallsetminus V(J)}.$$

Proof. (of Lemma B.0.2) The proof is a simple exercise in commutative algebra. One first proves (see [2, Exercise 1.12]) the following formula for a triple of ideals $I, J, K \subset R$:

$$(I : J + K) = (I : J) \cap (I : K),$$

from which it follows

$$((I \cdot J) : (I + J)) = ((I \cdot J) : I) \cap ((I \cdot J) : J) = I \cap J.$$

□

Remark B.0.4. Lemma B.0.2 describes the relationship between the schemes associated with the product and the intersection of two ideals (Equation (B.0.2)) and provides an inclusion-exclusion principle for subschemes (Equation (B.0.3)):

$$V(I \cap J) = \overline{V(I \cdot J) \smallsetminus V(I + J)} \tag{B.0.2}$$

$$V(I) \cup V(J) = \overline{V(I \cdot J) \smallsetminus (V(I) \cap V(J))} \tag{B.0.3}$$

Example B.0.5. Equation (B.0.3) in Remark B.0.4 justifies the fact that (see in Figure A.1) the exceptional divisor of X_2 in Appendix A has two irreducible components (respectively corresponding to I_L and I_M), while the exceptional divisor of X_4 has three irreducible components (two respectively corresponding to I_L and I_M, and the third corresponding to $I_L + I_M = I_p$).

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